

NOTES ON CONCRETE SHEAVES AND GENERATED SPACES  
VERSION 0.1      April 23, 2026

HAYATO NASU

This is a summary of what I have learned about **concrete sheaves** and convenient categories of spaces, mostly through multiple nLab entries and references on them. I found the general theory interesting and beautiful, but results seem scattered. I leave this article incomplete so that I can update it as I learn more, and it might be useful for others as well. Feedback, comments, and suggestions are very welcome.

Light grey remarks are less important and can be ignored on the first reading with no loss of continuity. I will cite some nLab entries, which is colored in green, and they are collected as a list at the end of this note.

0. ROUGH OVERVIEW

The categories of topological spaces, smooth manifolds, etc., do not show good categorical properties. For instance, the above two categories fail to be cartesian closed, and the latter does not even have all limits. There are several categorical machinery that we can use to circumvent such issues; they would fall into the two kinds of approaches: (1) extending the category of spaces by considering richer structures, and (2) restricting the category to a full subcategory with good properties.

**0.1. Two kinds of nice categories of spaces.** As for the first approach, one of the most beloved methods in category theory is to take the category of sheaves on the category we start with, and once you equip a **subcanonical coverage**, you can embed the original category as a full subcategory of the sheaf category. Roughly speaking, when you have a category of spaces with a coverage, whether it has a stricter notion of maps than continuous maps or not, the sheaves on that category can be seen as spaces obtained by freely glueing those spaces under the rule that the coverage prescribes.

A potential problem with this approach is that when you start with a large category, the resulting sheaf category could be too large to be a topos. For instance, the fundamental concept in condensed mathematics is **condensed sets**, which are defined as sheaves on the category of profinite spaces, but it suffers from the size issue, and there seem to have been different ideas to get along with it (light condensed sets, using different universes, etc.). Sheaves are considered nice because they form a topos when the underlying category is small, but a question can be raised whether we really need a topos, or all sheaves? Concrete sheaves are a good alternative to the full sheaf category, and even without the smallness assumption, they form a complete and cocomplete **quasitopos**, which may be decent enough to develop mathematics in. The philosophy behind concrete sheaves is that rather than glueing spaces freely in the air, you are only allowed to attach them onto a given set. Then, there is a size limit controlled by sets (or **Set**) so that the category of concrete sheaves forms nice properties. This idea dates back to Spanier's **quasitopologies** [Spa63], and is used to produce examples of **generalized smooth spaces** [BH11].

As for the second approach, there seems no canonical choice of how you construct that full subcategory. Imagine you know a class of spaces that you would like to include and make your category cocomplete. If you are a homotopy theorist, you may want to include all CW complexes, or if you only care about convergence of sequences, you may need just  $\mathbb{N}_\infty$  (the one-point compactification of the natural numbers). In any case, your first trial would be to take the colimit closure of that class (whose objects we call *spaces generated by that class*) in **Top**, and that is the minimal subcategory for your purpose. This naive approach turns out to yield categories of spaces with surprisingly desirable properties more than just cocompleteness, which is why this is a good resource of **convenient categories of spaces**. This includes **compactly generated spaces** and **sequential spaces**.

**0.2. The adjunction between concrete sheaves and generated spaces.** An intriguing aspect is how these two constructions fit together. Suppose you have a not necessarily full subcategory  $\mathcal{T}$  of **Top** with a coverage  $\mathbf{J}$ , and you can take the category  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  of concrete sheaves on it. Should

$$\begin{array}{ccccc}
& & \mathcal{T} & & \\
& \swarrow & \downarrow & \searrow & \\
\mathbf{CSh}(\mathcal{T}, \mathbf{J}) & \xleftrightarrow{\perp} & \mathbf{Top}_{\mathcal{T}} & \xleftrightarrow{\perp} & \mathbf{Top}
\end{array}$$

certain conditions be satisfied, there is an adjunction between this and the category  $\mathbf{Top}$  of topological spaces, whose fixed objects are exactly the spaces generated by  $\mathcal{T}$ . Based on this adjunction, we carry nice properties of the category of concrete sheaves, as a quasitopos, to the category of generated spaces as a reflective subcategory.

We have already showcased some examples of concrete sheaves and generated spaces, and [Figure 1](#) is a list of the examples we will discuss in this note.

Site of prototypes	concrete sheaves	The sheaf category	Generated spaces
$(\mathcal{T}, \mathbf{Triv})$	Concrete presheaves	$[\mathcal{T}^{\text{op}}, \mathbf{Set}]$	$\mathcal{T}$ -generated spaces
$(\mathbf{Mfd}, \mathbf{OpCov})$	Diffeological spaces	The topos of smooth sets	delta generated spaces
$(\mathbf{Cvx}, \mathbf{OpCov})$	Chen spaces	?	delta generated spaces
$(\{\mathbb{N}_{\infty}, 1\}, \mathbf{Canon})$	subsequential spaces (Kuratowski limit spaces)	Johnstone's topological topos	sequential spaces
$(\mathbf{CptHdf}, \mathbf{OpCov})$	Spanier's quasi-topological spaces	?	Compactly generated spaces (k-spaces, Kelley spaces)
$(\mathbf{CptHdf}, \mathbf{FinSurj})$	?	The category of condensed sets	Compactly generated spaces (k-spaces, Kelley spaces)

FIGURE 1. Examples of concrete sheaves and generated spaces (see [Section 1.6](#))

**0.3. Warm-up: two categorical gadgets.** Finally, I would like to mention two categorical gadgets that are useful for the discussion in this note, which might be less known to some readers.

*Topological categories.* One key feature of the category of topological spaces is that you can equip the final (finest) and initial (coarsest) topologies on a set with respect to a family of maps from or to some topological spaces. This is generalized by the notion of **topological categories** (topological functors into  $\mathbf{Set}$ ). Once you have a topological category, it is often the case that structures or properties of  $\mathbf{Set}$  can be lifted along the forgetful functor. We will use this powerful categorical tool to prove that concrete sheaves form a quasitopos, essentially following the argument in [\[DE06a\]](#). However, one should be able to avoid using this as well, as in [\[BH11\]](#).

*Weighted colimits for Kan extensions.* Let us take another look at final topologies. Given a set  $X$  and a family of maps  $\{f: S \rightarrow X\}$  from topological spaces in a fixed subcategory  $\mathcal{T}$ , and suppose you want to endow  $X$  with the final topology with respect to this family. This is more or less glueing the spaces  $S$  together along the maps  $f$  and forming it as a topological space. Gluing objects together is not a new idea in category theory; it is exactly what colimits do.

What kind of colimit is this? We have a candidate of a diagram to take the colimit of, which is the inclusion functor  $\mathcal{T} \rightarrow \mathbf{Top}$ , because we are patching spaces in  $\mathcal{T}$  together somehow, but in a way specified by the family of maps. **Weighted colimits** allow us to achieve this idea: for each space  $S$ , we take as many copies of  $S$  as the number of maps  $f: S \rightarrow X$  in the family, and then we glue them together with respect to the interactions between these maps in  $\mathcal{T}$ .

Let us be more precise. We define a weight  $W$  to be a presheaf  $\mathcal{T}^{\text{op}} \rightarrow \mathbf{Set}$ , which can be taken as a subpresheaf  $W \subseteq \mathbf{Top}(i_{\mathcal{T}}(-), X)$ , which defines the family of maps we want to glue together. Here,  $i_{\mathcal{T}}: \mathcal{T} \rightarrow \mathbf{Top}$  is the inclusion functor. Weighted cocones of  $i_{\mathcal{T}}$  with respect to this weight looks like [Figure 2](#) (compare with [Figure 3](#)). The weighted colimit is the universal such weighted cocone. Taking  $W$  as above, the weighted colimit is the set  $X$  equipped with the final topology with respect to  $W$ . For more details on weighted colimits in general, see [\[Per24, Section 3\]](#).

Weighted colimits are often used in the enriched setting, but even in the ordinary ( $\mathbf{Set}$ -enriched) case, it is a flexible way to directly describe what colimits we want to take, instead of combination of ordinary (conical) colimits. In particular, it is a canonical way to compute the value of a pointwise left Kan extension, meaning that if you have a pointwise Kan extension  $\mathbf{Lan}_F G$  for  $F: \mathcal{A} \rightarrow \mathcal{B}$  and  $G: \mathcal{A} \rightarrow \mathcal{C}$ , then  $\mathbf{Lan}_F G(B)$  is the weighted colimit of  $G$  with respect to the weight  $\mathcal{B}(F(-), B)$ . Other formulas for Kan extensions are also available as we work with ordinary categories, but the weighted

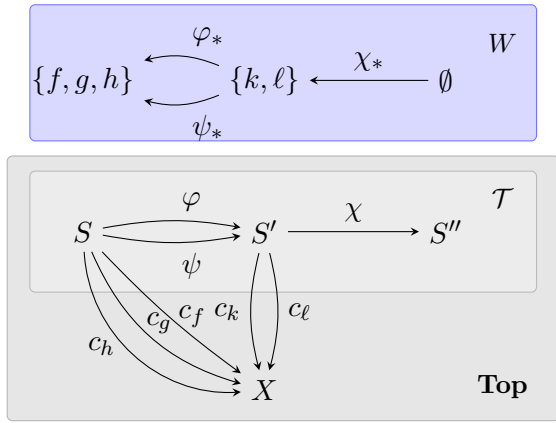


FIGURE 2. A cocone weighted by  $W$

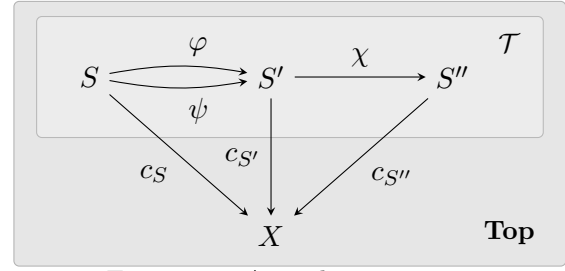


FIGURE 3. An ordinary cocone

colimit formula illustrates what kind of universal property the values of the Kan extension satisfy more accurately.

**0.4. Further reading and personal notes.** This is my personal compilation of the results scattered in the literature and nLab to understand them systematically. Here, I share some papers that I found nice to read and helpful to write this note, and also monumental papers in this area. This is by no means a comprehensive list.

First, we list a couple of introductory materials for topology in the categorical setting. The notes [Rie] is a good read that illuminate how constructions in topology can be understood in a categorical way, and the textbook [BBT20] is a nice introduction in a similar spirit. For topological categories, [AHS90] is a good reference. [Pre02] is a high-level textbook that has a lot of examples and cover a wide range of topics in categorical topology, including topological categories (the book calls them *topological constructs*).

Spanier’s paper [Spa63] seems the earliest paper to study quasitopological spaces, and it was observed in [Pen77] that the category of quasitopological spaces is an example of a quasitopos, which the author defined in that paper, and this perspective was further developed in [Wyl76]. The paper [DE06a] provides a more general framework for categories of concrete sheaves using the theory of topological categories. Modern accounts of concrete sheaves can be found in [BH11]. In Johnstone’s book, [Joh02a, A2.6] is a relevant section for quasitoposes in general, and [Joh02b, C2.2] for bisites and biseparated presheaves.

The adjunction between concrete sheaves and topological spaces is discussed on nLab (for instance, the entry on **diffeological space**). This adjunction is observed in [SYH18] and also in [CSW14]. However, seeing the category of spaces generated by a certain class of spaces as a reflective subcategory of separated presheaves can be traced back to Day’s paper on reflective subcategories [Day72].  $\mathcal{T}$ -generated spaces for a subcategory  $\mathcal{T}$  of **Top** have been studied in the context of homotopy theory, primarily in search of a convenient category of spaces [Ste67; Vog71; Wyl73].

I would also like to share some interesting papers and writings about a couple of specific examples of concrete sheaves or generated spaces.

- Diffeological spaces are a good example of concrete sheaves. There is a textbook [Igl13], and a shorter introduction [Igl21]. For a more category-theoretic account, see [BH11]. The corresponding generated spaces are called delta generated spaces, and they are studied in [Dug03; SYH18].
- The site consisting of the one-point compactification of the natural numbers  $\mathbb{N}_\infty$  and the one-point space 1 with the canonical coverage provides a categorical setting where convergence of sequences controls the spaces. [Har88] is good for a comprehensive account. I found the blog post [Gro24] quite helpful to understand related topics. This blog post [Sea14] and its following discussion are also nice to read.
- This paper [ELS04] discusses both the general theory of convenient categories of spaces and some specific examples, including compactly generated spaces and their analogs.

There are some topics studied in the literature that are not covered in this note, but I would like to mention them here to remind myself to look into them in the future.

- The model structures on the categories should be the next thing to look into. Textbooks on homotopy theory have a good account on the model structure on the category of (generated) spaces,<sup>1</sup> and on the category of diffeological spaces, they are studied in [HS24; Kih19].
- The whole discussion on quasispaces and generated spaces is generalized to  $(T, \mathcal{V})$ -categories in [Rib19].
- Some other generalizations of smooth manifolds, such as *Frölicher spaces* and differentiable spaces [CSW14] are not captured by concrete sheaves.

### Notations.

- We use  $\circ$  for composition in the conventional order, and  $\circledast$  for composition in the diagrammatic order, i.e.,  $g \circledast f = f \circ g$ . We will often use the former for functors, and the latter for morphisms in a category.
- For a presheaf  $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  and a morphism  $f: U \rightarrow V$  in  $\mathcal{C}$ , we write  $P(f)(x)$  as  $f \cdot x$  for  $x \in PV$ . Note that  $(g \circledast f) \cdot x = g \cdot (f \cdot x)$  holds.
- We will often work with a category  $\mathcal{C}$  with a terminal object  $1$ , and we write  $|U|$  for  $\mathcal{C}(1, U)$  for each object  $U$  in  $\mathcal{C}$ , which is occasionally called the *underlying set* of  $U$ .

## 1. CONCRETE SHEAVES AND QUASITOPPOSES

In this section, we summarize the theory of concrete sheaves for general concrete sites. In particular, we show that the category of concrete sheaves on a concrete site is topological over  $\mathbf{Set}$ , and is a complete and cocomplete quasitopos. In Section 1.6, we give examples of concrete sites and their concrete sheaves from the topological and smooth world.

### 1.1. Concrete presheaves.

**Remark 1.1 (Coverages, (Grothendieck) topologies).** There are multiple ways to define sites, and we fix the terminology here. *Coverages* refer to a collection of families of morphisms satisfying only the stability condition, whereas *(Grothendieck) topologies* refer to Lawvere-Tierney topologies on the presheaf category. A *site* is a pair of a (not necessarily small) category and a coverage on it. We use the first notion as a convenient presentation for sites, while the second notion is to determine when two sites are equivalent. Note that a site does not require pullbacks to exist in the underlying category, but in practice, the category  $\mathcal{C}$  usually has pullbacks that are needed for the stability condition.  $\lrcorner$

**Remark' 1.2.** Whatever people call a topology on a category, including the definition of coverages we adopt here, induces its category of sheaves, and two different topologies may induce the same category of sheaves. However, Lawvere-Tierney topologies on a presheaf category correspond bijectively to the sheaf subcategories of the presheaf category. That's why we want to see the equality of Grothendieck topologies as the correct notion of equivalence of sites, and we may deem Grothendieck topologies as equivalence classes of any notion of topologies. There is a concrete description of Grothendieck topologies in terms of sieves, but we do not use it in this note.  $\lrcorner$

**Definition 1.3 ([BH11, Definition 4.9]).** A (possibly large) site  $(\mathcal{C}, \mathbf{J})$  is said to be *concrete* if  $\mathcal{C}$  has a terminal object  $1$ ,  $|-| := \mathcal{C}(1, -): \mathcal{C} \rightarrow \mathbf{Set}$  is faithful, and for every covering family  $\{\iota_j: U_j \rightarrow U\}_{j \in J}$  in  $\mathbf{J}$ , the family of functions  $\{|\iota_j|: |U_j| \rightarrow |U|\}_{j \in J}$  is jointly surjective.  $\lrcorner$

When a category  $\mathcal{C}$  has a terminal object  $1$ , we write  $|U|$  for  $\mathcal{C}(1, U)$  as before.

**Definition 1.4 (Concrete presheaves).** Let  $\mathcal{C}$  be a category with a terminal object  $1$ . A *concrete presheaf*<sup>2</sup>  $P = (|P|, (PU)_{U \in \mathcal{C}})$  on  $\mathcal{C}$  consists of a set  $|P|$  with a family of subsets  $PU \subseteq |P|^{|U|}$  for each object  $U$  in  $\mathcal{C}$ , such that the following conditions hold:

<sup>1</sup>I will refrain from citing any, as there are many and I have not read them.

<sup>2</sup>In some literature, an equivalent notion is called *extensional presheaf* [RS99]. What we described here depends on the choice of subobjects of  $|P|^{|U|}$  for each object  $U$ , and hence not invariant under isomorphisms of presheaves, but we adopt this definition for simplicity.

- (1) for any  $U \xrightarrow{f} U'$  in  $\mathcal{C}$  and any  $|U| \xrightarrow{p} |P|$  in  $PU$ , the composite  $|U| \xrightarrow{|f|} |U'| \xrightarrow{p} |P|$  belongs to  $PU'$ , and  
 (2) for any  $p \in |P|$ , the corresponding function  $|1| \xrightarrow{p} |P|$  belongs to  $P1$ .  $\lrcorner$

A concrete presheaf  $(|P|, (PU)_{U \in \mathcal{C}})$  certainly defines a presheaf  $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  by  $PU$  for each object  $U$  and the restriction maps given by precomposition, so we will just write  $P$  for a concrete presheaf. Morphisms between concrete presheaves are defined as morphisms between the presheaves they define. Explicitly, a morphism  $f: P \rightarrow Q$  between concrete presheaves is a function  $|f|: |P| \rightarrow |Q|$  such that for any  $U$  and any  $p \in PU$ , the composite  $|U| \xrightarrow{p} |P| \xrightarrow{|f|} |Q|$  belongs to  $QU$ . We denote the category of concrete presheaves on  $\mathcal{C}$  by  $\mathbf{CPSh}(\mathcal{C})$ .

**Remark 1.5.** Our definition of concrete presheaves is not invariant under isomorphisms of presheaves. We could modify the definition to be invariant by defining a concrete presheaf as a presheaf  $P$  such that for each object  $U$ , the canonical map

$$\gamma_U: PU \rightarrow (P1)^{|U|}; \quad p \mapsto (x \mapsto x \cdot p). \quad (1.1)$$

is injective, not an inclusion of a subset. Then, by defining  $|P|$  to be  $P1$  and replacing  $PU$  with the image of  $\gamma_U$ , we will have that every presheaf with this property is isomorphic to a concrete presheaf in the sense of Definition 1.4, and the category of these extended concrete presheaves is equivalent to  $\mathbf{CPSh}(\mathcal{C})$ .  $\lrcorner$

**Remark 1.6.** Definition 1.4(2) could be omitted from the definition, and some properties still hold in the category of those extended concrete presheaves. See [Wyl76, Section 2] for more details.<sup>3</sup>  $\lrcorner$

**1.2. Concrete sheaves on a site of prototypes.** Let us review the definition of sheaves on a site<sup>4</sup>.

**Definition 1.7 (Sheaves).** Let  $(\mathcal{C}, \mathcal{J})$  be a site. A presheaf  $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  is *a sheaf with respect to  $\mathcal{J}$  (or  $\mathcal{J}$ -sheaf)* if for any covering family  $\{\iota_i: U_i \rightarrow U\}_{i \in I}$ , the following canonical map is a bijection:

$$PU \rightarrow P(\{\iota_i\}_{i \in I}) := \left\{ (p_i)_{i \in I} \in \prod_{i \in I} PU_i \left| \begin{array}{l} k \cdot p_i = \ell \cdot p_j \text{ for any } V \\ \begin{array}{ccccc} & & U_i & & \\ & k \nearrow & \circ & \searrow \iota_i & \\ & \ell \searrow & U_j & \nearrow \iota_j & U \\ & & & & \end{array} & \text{in } \mathcal{C}. \end{array} \right. \right\}. \quad (1.2)$$

$$p \mapsto (\iota_i \cdot p)_{i \in I}$$

A presheaf is *separated with respect to  $\mathcal{J}$  (or  $\mathcal{J}$ -separated)* if the function (1.2) is injective for any covering family  $\{\iota_i: U_i \rightarrow U\}_{i \in I}$ .  $\lrcorner$

**Definition 1.8 (Concrete sheaves [Dub79, Definition 1.4]).** Let  $(\mathcal{C}, \mathcal{J})$  be a concrete site. A *concrete sheaf* on  $(\mathcal{C}, \mathcal{J})$  is a concrete presheaf  $P$  on  $\mathcal{C}$  which is a  $\mathcal{J}$ -sheaf. We denote the category of concrete sheaves on  $(\mathcal{C}, \mathcal{J})$  by  $\mathbf{CSh}(\mathcal{C}, \mathcal{J})$ .  $\lrcorner$

**Example 1.9.** Examples are abundant in subcategories of  $\mathbf{Top}$ , which we will discuss in more detail in Section 1.6. Here, we only present one of them, and a different example not necessarily seen as a subcategory of  $\mathbf{Top}$ .

- (1) The category of smooth manifolds  $\mathbf{Mfd}$  with the open cover topology is a concrete site. Concrete sheaves on it are called *diffeological spaces*. See Example 1.57(2) for more details.
- (2) The category of nonempty finite sets  $\mathbf{FinSet}_{\neq \emptyset}$  with the trivial topology is a concrete site. Concrete sheaves on it are just concrete presheaves on that category, and they are identified with *abstract simplicial complexes*. See [BH11, Proposition 4.18].
- (3) The category of standard Borel spaces  $\mathbf{StBS}$  (the full subcategory of measurable spaces consisting of the retracts of the measurable space  $\mathbb{R}$  with the Borel  $\sigma$ -algebra) with the countably

<sup>3</sup>In that paper, concrete presheaves are called *P-sieves*. According to that paper, the definition originally given by Antoine.

<sup>4</sup>I will stick to the very concrete definition here, but there are fancier ways to define them.

extensive coverage is a concrete site. Concrete sheaves on it are called **quasi-Borel spaces**, introduced in [Heu+17]. As they are not topological spaces, the argument in Section 2 does not apply, but by replacing **Top** with the category of measurable spaces, we expect the adjunction in the paper should be recasted by the nerve-and-realization adjunction.  $\lrcorner$

**Remark 1.10.** This idea of extending the notion of spaces by defining topological structures on sets by all “admissible maps” (this could mean either continuous maps, smooth maps, etc.) with some glueing property dates back to the work by Spanier [Spa63], where the structures are called *quasitopologies*, and used in Day’s paper [Day72].

Note that some authors do not require Definition 1.4 (2) for concrete sheaves, as in [Dub79], where the author used the term *quasispaces* for them. Then, the category of quasispaces is not realized as a full subcategory of the presheaf category, but it is still shown to be a quasitopos. In [DE06a], the corresponding notion to what we call concrete sheaves here is called *strict quasispaces*, but in the setting of topological categories over a general base.  $\lrcorner$

**Proposition 1.11 (Concrete description of concrete sheaves).** Let  $(\mathcal{C}, J)$  be a concrete site. A concrete presheaf  $P$  is a concrete  $J$ -sheaf if and only if it satisfies the following glueing condition:

$$\forall \{ \iota_i : U_i \rightarrow U \}_{i \in I} : J\text{-cov}, \forall p \in |P|^{|U|} \quad ( p \in PU \iff \forall i \in I. |\iota_i| \circ p \in PU_i ) \quad (1.3)$$

*Proof.* The canonical function (1.2) fits into the following diagram by the naturality:

$$\begin{array}{ccc} PU & \xrightarrow{(1.2)} & P(\{ \iota_i \}_{i \in I}) \\ \downarrow & & \downarrow \\ |P|^{|U|} & \xrightarrow{(\iota_i \circ -)_{i \in I}} & \prod_{i \in I} |P|^{|U_i|} \end{array}$$

The function at the bottom is injective, as the family  $\{ |\iota_i| : |U_i| \rightarrow |U| \}_{i \in I}$  is jointly surjective<sup>5</sup>, so (1.2) is always injective. If (1.3) holds, then any  $(p_i)_{i \in I} \in P(\{ \iota_i \}_{i \in I}) \subseteq \prod_{i \in I} PU_i$  is in the function at the bottom, so (1.2) is surjective, and thus  $P$  is a concrete sheaf. Conversely, if (1.2) is bijective, then any  $p \in |P|^{|U|}$  such that  $|\iota_i| \circ p \in PU_i$  for all  $i \in I$  belongs not only to  $\prod_{i \in I} PU_i$  but also to  $P(\{ \iota_i \}_{i \in I})$ , so it comes from  $p \in PU$ .  $\square$

The functions in  $PU$  are called *plots* in the context of diffeological spaces, and also called *admissible maps* in other contexts.

As we will see later in Section 3.1, the category  $\mathbf{CSh}(\mathcal{C}, J)$  of concrete sheaves can be seen as the category of biseparated presheaves on a bisite, under a mild assumption on  $\mathcal{C}$ . If  $\mathcal{C}$  is small, then such a category of biseparated presheaves is a Grothendieck topos. In particular, it is complete, cocomplete, locally cartesian closed, and has a small generator. However, we do not assume the smallness of  $\mathcal{C}$  in general, so we cannot conclude that  $\mathbf{CSh}(\mathcal{C}, J)$  is a quasitopos from this perspective.

Fundamental results on sheaf categories do not apply directly to our general setting because the site is not necessarily small. We always need to be careful about size issues. First, we show that  $\mathbf{CSh}(\mathcal{C}, J)$  is reflective in the presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ .

**Proposition 1.12 (Concrete-sheafification).** Let  $(\mathcal{C}, J)$  be a concrete site. The category  $\mathbf{CSh}(\mathcal{C}, J)$  is reflective in the presheaf category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ :

$$\mathbf{CSh}(\mathcal{C}, J) \xleftarrow[\perp]{L} [\mathcal{C}^{\text{op}}, \mathbf{Set}] .$$

*Proof.* We have the following sequence of left adjoints:

$$\mathbf{CSh}(\mathcal{C}, J) \xleftarrow[\perp]{L_1} \mathbf{CPSH}(\mathcal{C}) \xleftarrow[\perp]{L_0} [\mathcal{C}^{\text{op}}, \mathbf{Set}] .$$

<sup>5</sup>In terms of the bisite  $(\mathcal{T}, J, \text{Surj})$ , the  $\text{Surj}$ -separatedness with entails the  $J$ -separatedness with respect to  $J$ .



any lift  $\{k_i: Y_i \rightarrow Z\}_{i \in I}$  of  $\{f_i \circ h: UY_i \rightarrow UZ\}_{i \in I}$ , there exists a unique morphism  $\bar{h}: Y \rightarrow Z$  in  $\mathcal{D}$  such that  $U\bar{h} = h$  and  $\ell_i \circ \bar{h} = k_i$  for all  $i \in I$ .

$$\begin{array}{ccc}
 & \{k_i\}_i & \\
 & \curvearrowright & \\
 \{Y_i\}_i & \xrightarrow{\{\ell_i\}_i} & Y \xrightarrow{\exists! \bar{h}} Z & \text{in } \mathcal{D}, \\
 & \downarrow U & \\
 \{UY_i\}_i & \xrightarrow{\{f_i\}_i} & X \xrightarrow{h} UZ & \text{in } \mathcal{S}.
 \end{array}$$

We simply say that a sink is final if it is a final lift of the  $U$ -sink obtained by applying  $U$ .

A functor  $U: \mathcal{D} \rightarrow \mathcal{S}$  is called **topological** if every  $U$ -sink has a final lift.

Dually, we define **sources**  $\{f_i: X \rightarrow X_i\}_{i \in I}$ ,  **$U$ -sources**, **lifts of  $U$ -sources**, and **initial lifts of  $U$ -sources**.  $\lrcorner$

When a functor  $U: \mathcal{D} \rightarrow \mathcal{S}$  is obvious from the context, like “the” forgetful functor from a category to **Set**, the category  $\mathcal{D}$  is often called *topological over  $\mathcal{S}$*  or a *topological category*.

Requiring a functor to be topological is strong enough to imply the following conditions:

- $U^{\text{op}}$  is also topological (the duality theorem for topological functors),
- $U$  is a fibration (hence it is a bifibration by the duality),
- $U$  is faithful,
- the fibers  $\mathcal{D}_X$  are preordered,
- $U$  has both a left adjoint and a right adjoint, which are both fully faithful.

We could also split this condition into two parts: final lifts of unary sinks and final lifts of sinks consisting of identity morphisms, which gives another characterization of topological functors.

**Fact 1.17** ([DE06b, Proposition 4.3]). Let  $U: \mathcal{D} \rightarrow \mathcal{S}$  be a functor.  $U$  is topological if and only if it satisfies the following two conditions:

- (1) It is an fibration.
- (2) The fibers  $\mathcal{D}_X$  have all (possibly large) products for all objects  $X$  in  $\mathcal{S}$ , which are preserved by  $u^*: \mathcal{D}_Y \rightarrow \mathcal{D}_X$  for every morphism  $u: X \rightarrow Y$  in  $\mathcal{S}$ , where  $u^*$  is the reindexing functor given by the opfibration structure.  $\lrcorner$

**Corollary 1.18.** A functor  $U: \mathcal{D} \rightarrow \mathcal{S}$  is topological if and only if it satisfies the following conditions:

- (1)  $U$  is a bifibration, i.e., it is both a fibration and an opfibration.
- (2) Every fiber  $\mathcal{D}_X$  over an object  $X$  in  $\mathcal{S}$  has all (possibly large) products.  $\lrcorner$

*Proof.* If  $U$  is topological, then its opposite  $U^{\text{op}}$  is also topological, and thus  $U$  is an opfibration, so “only if” part holds. Conversely, we need to prove if  $U$  is an opfibration, the products in the fibers are stable under reindexing along morphisms in  $\mathcal{S}$ . However, under the assumption that  $U$  is a bifibration, we have the adjunctions  $u_* \dashv u^*$  for every morphism  $u: X \rightarrow Y$  in  $\mathcal{S}$ . In particular,  $u^*$  as a right adjoint preserves all limits, so the “if” part holds.  $\square$

**Remark 1.19.** By the dual argument, we can also characterize topological functors as bifibrations whose fibers have all (possibly large) coproducts. This is an analogue of the fact that a complete lattice is equivalent to a cocomplete lattice.  $\lrcorner$

**Proposition 1.20 (Limits/colimits in topological categories).** Let  $U: \mathcal{D} \rightarrow \mathcal{S}$  be a topological functor.

- (1) Suppose  $\mathcal{S}$  has limits of shape  $\mathcal{J}$  for a small category  $\mathcal{J}$ . Then,  $\mathcal{D}$  also has limits of shape  $\mathcal{J}$ , and  $U$  preserves them. The limit in  $\mathcal{D}$  of a diagram  $F: \mathcal{J} \rightarrow \mathcal{D}$  is given by the initial lift of the limit cone as a  $U$ -source.
- (2) Suppose  $\mathcal{S}$  has colimits of shape  $\mathcal{J}$  for a small category  $\mathcal{J}$ . Then,  $\mathcal{D}$  also has colimits of shape  $\mathcal{J}$ , and  $U$  preserves them. The colimit in  $\mathcal{D}$  of a diagram  $F: \mathcal{J} \rightarrow \mathcal{D}$  is given by the final lift of the colimit cocone as a  $U$ -sink.

In particular, when  $\mathcal{S}$  is complete and cocomplete, so is  $\mathcal{D}$ .  $\lrcorner$

**Proposition 1.21 (Monomorphisms/epimorphisms in topological categories).** Let  $U: \mathcal{D} \rightarrow \mathcal{S}$  be a topological functor, and suppose  $\mathcal{S}$  admits finite limits and finite colimits.

- (1) A morphism  $f: X \rightarrow Y$  in  $\mathcal{D}$  is a monomorphism (resp. epimorphism) if and only if  $Uf$  is a monomorphism (resp. epimorphism) in  $\mathcal{S}$ .
- (2) A morphism  $f: X \rightarrow Y$  in  $\mathcal{D}$  is strong monomorphism (resp. strong epimorphism) if and only if  $Uf$  is a strong monomorphism (resp. strong epimorphism) in  $\mathcal{S}$  and  $f$  is initial (resp. final) as a source (resp. sink) consisting of a single morphism.
- (3) A morphism  $f: X \rightarrow Y$  in  $\mathcal{D}$  is regular monomorphism (resp. regular epimorphism) if and only if  $Uf$  is a regular monomorphism (resp. regular epimorphism) in  $\mathcal{S}$  and  $f$  is initial (resp. final) as a source (resp. sink) consisting of a single morphism.  $\lrcorner$

In particular, when strong epimorphisms (resp. monomorphisms) coincide with regular epimorphisms (resp. monomorphisms) in  $\mathcal{S}$ , they also coincide in  $\mathcal{D}$ .

*Proof.* Because of the duality, we only need to prove the monomorphism part of each item.

- (1) The “only if” part follows from  $U$  being a right adjoint, and the “if” part follows from  $U$  being faithful.
- (2) For the “only if” part, preserving strong monomorphisms follows from  $U$  being a right adjoint. Suppose  $f$  is a strong monomorphism and we have the following diagram on the left:

$$\begin{array}{ccc}
 & & h \\
 & \curvearrowright & \\
 Z & & X \xrightarrow{f} Y \\
 & \curvearrowleft & \\
 & & \\
 Uh & \downarrow U & \\
 UZ \xrightarrow{k} UX & \xrightarrow{Uf} & UY \\
 & & \\
 & & \text{in } \mathcal{S}.
 \end{array}
 \quad \text{in } \mathcal{D},
 \quad \begin{array}{ccc}
 \Delta UZ & \xrightarrow{k'} & X \\
 \varepsilon_Z \downarrow & \nearrow & \downarrow f \\
 Z & \xrightarrow{h} & Y \\
 & \exists! \bar{k} & \\
 & & \text{in } \mathcal{S}.
 \end{array}$$

Let  $\Delta$  be the left adjoint of  $U$ ,  $\varepsilon$  be the counit of the adjunction, and  $k'$  be the morphism corresponding to  $U$ . Then, the diagram on the right commutes because of the adjunction  $\Delta \dashv U$ . Since  $\Delta$  is fully faithful, the morphism  $U\varepsilon_Z$  is an isomorphism, so  $\varepsilon_Z$  is an epimorphism by [Proposition 1.21 \(1\)](#). Therefore, by the definition of strong monomorphisms, there exists a unique morphism  $\bar{k}$  that makes the diagram on the right commute. It is easy to see this  $\bar{k}$  is the desired morphism. For the “if” part, if we have a commutative square for the orthogonality condition in  $\mathcal{D}$ , then sending it by  $U$  gives a unique filler in  $\mathcal{S}$  because  $Uf$  is a strong monomorphism, and one can lift it to a morphism in  $\mathcal{D}$  because  $f$  is initial.

- (3) Both conditions entail [Proposition 1.21 \(2\)](#), so we can assume  $f$  is initial. The “only if” part follows from  $U$  being a right adjoint, and the “if” part follows from  $U$  lifts limits in the sense of [Proposition 1.20](#).  $\square$

**Definition 1.22.** A sink  $\{f_i: X_i \rightarrow X\}_{i \in I}$  in a category  $\mathcal{S}$  is called *effectively epimorphic*<sup>7</sup> if for another sink  $\{g_i: X_i \rightarrow Y\}_{i \in I}$  that satisfies the following condition called *compatibility* with respect to  $\{f_i: X_i \rightarrow X\}_{i \in I}$ , there exists a unique morphism  $h: X \rightarrow Y$  such that  $g_i = f_i \circ h$  for all  $i \in I$ :

$$\begin{array}{ccc}
 \text{For any } i, j \in I \\
 \text{and } Z \xrightarrow{\ell} X_i, Z \xrightarrow{m} X_j, & \begin{array}{ccc} & X_i & \\ \ell \nearrow & \circ & \searrow f_i \\ Z & & X \\ m \searrow & & \nearrow f_j \end{array} & \implies & \begin{array}{ccc} & X_i & \\ \ell \nearrow & \circ & \searrow g_i \\ Z & & Y \\ m \searrow & & \nearrow g_j \end{array}
 \end{array}$$

$\lrcorner$

When kernel pairs exist, one can rephrase this using them. In particular, a sink consisting of a single morphism is effective epimorphic if and only if the morphism is a regular epimorphism under this assumption.

**Definition 1.23.** A category admits *regular factorizations of sinks* if every sink  $\{f_i: X_i \rightarrow X\}_{i \in I}$  can be factored as a effective epimorphic sink followed by a monomorphism.  $\lrcorner$

<sup>7</sup>This is also called *strict epimorphic*.

Analogous to *orthogonal factorization systems*, there is a theory of factorization structures for sinks and sources in categories, but we do not go into the details here.

**Example 1.24.** In **Set**, effective epimorphic sinks are exactly the jointly surjective sinks. **Set** admits regular factorizations of sinks by taking the union of the images of the morphisms in the sink.  $\lrcorner$

**Proposition 1.25** ([DE06b, Theorem 2.17]). Let  $U: \mathcal{D} \rightarrow \mathcal{S}$  be a topological functor. A sink  $\{f_i: X_i \rightarrow X\}_{i \in I}$  in  $\mathcal{D}$  is effectively epimorphic if and only if the following two conditions hold:

- (1) The sink  $\{Uf_i: UX_i \rightarrow UX\}_{i \in I}$  in  $\mathcal{S}$  is effectively epimorphic.
- (2) The sink  $\{f_i: X_i \rightarrow X\}_{i \in I}$  is final as a  $U$ -sink.  $\lrcorner$

In the literature, when  $\mathcal{S}$  is **Set** or something similar, sinks satisfying the above two conditions are sometimes said to be *final surjective*.

*Proof.* The proof is almost the same as Proposition 1.21, but for arbitrary sinks instead of unary ones.  $\square$

**Definition 1.26.** Let  $\mathbf{E}$  be a class of sinks in a category  $\mathcal{S}$ .

- (1) Suppose  $\mathcal{S}$  has binary products.  $\mathbf{E}$  is said to be *simply stable* if for every sink  $\{f_i: X_i \rightarrow X\}_{i \in I}$  in  $\mathbf{E}$  and every object  $Y$  in  $\mathcal{S}$ , the sink  $\{f_i \times \text{id}_Y: X_i \times Y \rightarrow X \times Y\}_{i \in I}$  is also in  $\mathbf{E}$ .
- (2) Suppose  $\mathcal{S}$  has pullbacks.  $\mathbf{E}$  is said to be (*pullback*) *stable*<sup>8</sup> if for every sink  $\{f_i: X_i \rightarrow X\}_{i \in I}$  in  $\mathbf{E}$  and every morphism  $u: Y \rightarrow X$  in  $\mathcal{S}$ , the sink  $\{p_i: X_i \times_X Y \rightarrow Y\}_{i \in I}$  obtained as the pullbacks of  $f_i$  along  $u$  is also in  $\mathbf{E}$ .  $\lrcorner$

Now we mention lifting of adjunctions along topological functors, which is a variation of a theorem called *Weyler's taut adjoint lifting theorem* in [Wy171].

**Proposition 1.27 (Adjoint triangle theorem for topological functors).** Suppose we have the following diagram in **CAT** where  $V$  is a topological functor,  $\mathcal{D}$  admits regular factorizations of sinks, and  $F$  preserves effective epimorphic sinks:

$$\mathcal{D} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{E} \xrightarrow{V} \mathcal{B}$$

Then, a functor  $F$  has a right adjoint if and only if  $V \circ F$  has a right adjoint.  $\lrcorner$

*Proof.* As any topological functor has a right adjoint, the “only if” part is straightforward. For the “if” part, let  $G'$  be the right adjoint of  $V \circ F$ , and define  $G'X$  for an object  $X$  in  $\mathcal{E}$  as follows: Take the family  $\{f: FY_f \rightarrow X\}$  consisting of all morphisms in  $\mathcal{E}$  of this form, and apply the following procedure to it:

$$\begin{aligned} \{FY_f \xrightarrow{f} X\}_f \text{ in } \mathcal{E} &\xrightarrow{\text{applying } V} \{V FY_f \xrightarrow{Vf} VX\}_f \text{ in } \mathcal{B} \\ &\xrightarrow{\text{by } V \circ F \dashv G'} \{Y_f \xrightarrow{\overline{Vf}} G' VX\}_f \text{ in } \mathcal{D} \\ &\xrightarrow{\text{taking the regular factorization in } \mathcal{D}} \{Y_f \xrightarrow{e_f} GX\}_f \text{ in } \mathcal{D} \end{aligned}$$

By construction, we have a monomorphism  $m: GX \rightarrow G'VX$  in  $\mathcal{D}$  such that  $e_f \circ m = \overline{Vf}$  for all  $f$ . We aim to show that  $G'X$  is the value of the right adjoint  $G$  at  $X$ .

We have  $VFe_f \circ \overline{m} = Vf$  for all  $f$ , because  $e_f \circ m = \overline{Vf}$ , so we have the following diagram:

$$\begin{array}{ccc} & \{f\}_f & \\ & \curvearrowright & \\ \{FY_f\}_f & \xrightarrow{\{Fe_f\}_f} & FGX \xrightarrow{\exists! v_X} X & \mathcal{E} \\ & \downarrow V & & \downarrow V \\ \{V FY_f\}_f & \xrightarrow{\{VFe_f\}_f} & VFGX \xrightarrow{\overline{m}} VX & \mathcal{B} \end{array}$$

<sup>8</sup>In the papers [DE06b; DE06a] and similar contexts, this is called *universal*.

By the assumption, the sink  $\{Fe_f: FY_f \rightarrow FGX\}_f$  is effective epimorphic, and by [Proposition 1.25](#), it is final as a  $V$ -sink. Therefore, there exists a unique morphism  $v_X: FGX \rightarrow X$  such that  $f = Fe_f \circ v_X$  for all  $f$ .

Now we have half of the universal property of  $v_X$ , as every  $f$  factors as  $f = Fe_f \circ v_X$ . Suppose we have a morphism  $h: Y \rightarrow GX$  such that  $f = Fh \circ v_X$ . Then,  $Vf = VFh \circ \overline{m}$ , and thus  $Vf$  corresponds to  $h \circ m$  under the adjunction  $V \circ F \dashv M$ . However,  $m$  is a monomorphism, so such  $h$  is uniquely determined, namely  $h = e_f$ . This shows the other half of the universal property, so  $GX$  is the value of the right adjoint  $G$  at  $X$ .  $\square$

**Proposition 1.28 (Adjoint lifting along topological functors (cf. [\[DE06a, Theorem 2.10\]](#))).** Suppose we have the following diagram in **CAT** where  $U$  and  $V$  are topological functors and  $V \circ F = L \circ U$ :

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{E} \\ U \downarrow & & \downarrow V \\ \mathcal{S} & \xrightarrow{L} & \mathcal{B} \\ & \xleftarrow[R]{\perp} & \end{array}$$

Furthermore, suppose that  $\mathcal{S}$  admits regular factorizations of sinks and that  $F$  preserves effective epimorphic sinks. Then,  $F$  has a right adjoint  $G: \mathcal{E} \rightarrow \mathcal{D}$ .  $\lrcorner$

*Proof.*  $U$  has a right adjoint as it is a topological functor, and so is  $L \circ U$ . Therefore, to apply [Proposition 1.27](#), it suffices to show that  $\mathcal{D}$  admits regular factorizations of sinks. Let  $\{f_i: X_i \rightarrow X\}_{i \in I}$  be a sink in  $\mathcal{D}$ . By the assumption, we have a factorization of the  $U$ -sink  $\{Uf_i: UX_i \rightarrow UX\}_{i \in I}$  as  $\{e_i: UX_i \rightarrow Y\}_{i \in I}$  followed by  $m: Y \rightarrow UX$ . Now take the final lift of  $\{e_i: UX_i \rightarrow Y\}_{i \in I}$  to obtain a sink  $\{\ell_i: X_i \rightarrow Z\}_{i \in I}$  in  $\mathcal{D}$  such that  $U\ell_i = e_i$  for all  $i$ . By [Proposition 1.25](#), this is an effective epimorphic sink in  $\mathcal{D}$ . Since it is final, there exists a unique morphism  $n: Z \rightarrow X$  such that  $Un = m$  and  $\ell_i \circ n = f_i$  for all  $i$ . By [Proposition 1.21 \(1\)](#),  $n$  is a monomorphism, so we have the desired factorization of  $\{f_i: X_i \rightarrow X\}_{i \in I}$ .  $\square$

**Remark' 1.29.** For  $G$  to satisfy  $U \circ G = R \circ V$ , we need additional assumptions that  $F$  preserves all final sinks. In that case, the factorization condition is not necessary, see [\[AH86, Theorem I.7\]](#) for the special case of this statement when  $F$  is of the form  $- \times X$ .  $\lrcorner$

**Lemma 1.30.** Let  $U: \mathcal{D} \rightarrow \mathcal{S}$  be a topological functor.

- (1) For each  $X$  in  $\mathcal{D}$ , the functor  $U/X: \mathcal{D}/X \rightarrow \mathcal{S}/UX$  induced on the slice categories is also topological. Final lifts of  $U/X$ -sinks underlie final lifts of the underlying  $U$ -sinks. In particular, a sink in  $\mathcal{D}/X$  is effective epimorphic if and only if its underlying sink in  $\mathcal{D}$  is effective epimorphic.
- (2) If  $\mathcal{S}$  admits regular factorizations of sinks, then so does  $\mathcal{S}/A$  for every object  $A$  in  $\mathcal{S}$ , which is preserved by the projection functor  $\mathcal{S}/A \rightarrow \mathcal{S}$ .  $\lrcorner$

*Proof.* Follows from the fact that  $\mathcal{D}/X \rightarrow \mathcal{D}$  creates all colimits and connected limits for every object  $X$  in  $\mathcal{D}$ .  $\square$

**Corollary 1.31 (When topological functors lift (local) cartesian closure).** Let  $U: \mathcal{D} \rightarrow \mathcal{S}$  be a topological functor where  $\mathcal{S}$  has finite limits and admits regular factorizations of sinks.

- (1) If  $\mathcal{S}$  is cartesian closed and effectively epimorphic sinks in  $\mathcal{D}$  are simply stable, then  $\mathcal{D}$  is cartesian closed.
- (2) If  $\mathcal{S}$  is locally cartesian closed and effectively epimorphic sinks in  $\mathcal{D}$  are stable, then  $\mathcal{D}$  is locally cartesian closed.  $\lrcorner$

*Proof.* (1) We apply [Proposition 1.28](#) to the functor  $- \times X: \mathcal{D} \rightarrow \mathcal{D}$  over  $- \times UX: \mathcal{S} \rightarrow \mathcal{S}$  for each object  $X$  in  $\mathcal{D}$ . The simply stability of effectively epimorphic sinks in  $\mathcal{D}$  ensures that  $- \times X$  preserves effective epimorphic sinks in  $\mathcal{D}$ .

- (2) We apply [Proposition 1.28](#) to the functor  $- \times_X Y: \mathcal{D}/X \rightarrow \mathcal{D}/X$  over  $- \times_{UX} UY: \mathcal{S}/UX \rightarrow \mathcal{S}/UX$  for each morphism  $Y \rightarrow X$  in  $\mathcal{D}$ , which is possible by [Lemma 1.30](#). The stability of effectively

epimorphic sinks in  $\mathcal{D}$  and the description of effective epimorphic sinks in  $\mathcal{D}/X$  ensures that  $- \times_X Y$  preserves effective epimorphic sinks in  $\mathcal{D}/X$ .  $\square$

**Remark 1.32.** Exponentials  $Y^X$  in  $\mathcal{D}$  (resp. in  $\mathcal{D}/X$ ) are not preserved by  $U$  in general. When  $\mathcal{S}$  is **Set**, direct calculation shows that  $Y^X$  underlies the set of all functions from  $|X|$  to  $|Y|$  coming from morphisms in  $\mathcal{D}$ . For instance, when  $\mathcal{D}$  is the category of topological spaces,  $Y^X$  underlies the set of all continuous functions from  $|X|$  to  $|Y|$ , which is a subset of the set of all functions from  $|X|$  to  $|Y|$ . For  $U$  to preserve exponentials, we need extra conditions, as inferred from [Remark 1.29](#).  $\lrcorner$

**1.4. The category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  of concrete sheaves is topological.** The goal of this part is to prove the following theorem.

**Theorem 1.33.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The forgetful functor  $|-|: \mathbf{CSh}(\mathcal{C}, \mathbf{J}) \rightarrow \mathbf{Set}$  is topological.  $\lrcorner$

**Lemma 1.34.** Let  $\mathcal{C}$  be a category with a terminal object  $1$  and  $(-)(1): [\mathcal{C}^{\text{op}}, \mathbf{Set}] \rightarrow \mathbf{Set}$  be the evaluation functor at  $1$ . Then  $(-)(1)$  is a bifibration.  $\lrcorner$

*Proof.* In general, any injective-on-object fully-faithful functor  $F: \mathcal{C}' \rightarrow \mathcal{C}$  induces a bifibration  $F^*$  from  $[\mathcal{C}, \mathbf{Set}]$  to  $[\mathcal{C}', \mathbf{Set}]$  as long as Kan extensions along  $F$  exist. Here, the construction of cartesian (resp. opcartesian) lifts involves pullbacks (resp. pushouts) and right (resp. left) Kan extensions. We do not give the detailed proof of this statement, but just give the construction of cartesian and opcartesian lifts in our specific case.

Given a morphism  $f: X \rightarrow Y$  in **Set** and an object  $B: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  with  $UB = Y$ , we define a presheaf  $A: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  by

$$\begin{array}{ccc} Ac & \longrightarrow & [\mathcal{C}(1, c), X] \\ \ell_c =: \downarrow & \lrcorner & \downarrow f_* \\ Bc & \xrightarrow{b \mapsto b \cdot -} & [\mathcal{C}(1, c), Y] \end{array}$$

Then,  $\ell: A \rightarrow B$  is a cartesian lift of  $f$  at  $B$ .

On the other hand, given a morphism  $f: X \rightarrow Y$  in **Set** and an object  $A: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  with  $UA = X$ , we define a presheaf  $B: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  by

$$\begin{array}{ccc} X & \xrightarrow{A(!)} & Ac \\ f \downarrow & \lrcorner & \downarrow := k_c \\ Y & \longrightarrow & Bc \end{array}$$

$\square$

Now we discuss when a “reflective subfunctor” of a fibration (resp. opfibration) is again a fibration (resp. opfibration).

**Proposition 1.35.** Suppose we have a fully faithful inclusion in  $\mathbf{CAT}/\mathcal{C}$ , which has a left adjoint in this 2-category:

$$\begin{array}{ccc} \mathcal{D}' & \xleftarrow{L} & \mathcal{D} \\ & \lrcorner & \\ & F' \searrow & \swarrow F \\ & & \mathcal{C} \end{array} .$$

- (1) If  $F$  is a fibration and cartesian lifts are closed under the inclusion  $\mathcal{D}' \hookrightarrow \mathcal{D}$ , that is, for every cartesian lift of  $u: x \rightarrow Fd$  where  $d$  is in  $\mathcal{D}'$ , the domain of the lift is also in  $\mathcal{D}'$ , then  $F'$  is a fibration.
- (2) If  $F$  is an opfibration, then  $F'$  is an opfibration where opcartesian lifts are given by composing the opcartesian lifts in  $\mathcal{D}$  with the unit of the adjunction.  $\lrcorner$

*Proof.*

- (1) This is straightforward from the assumption as we have the  $\mathcal{D}'$ -reflection of each object in  $\mathcal{D}$  that lies over the same object in  $\mathcal{C}$ .
- (2) Given a morphism  $f: FY \rightarrow X$  in  $\mathcal{C}$  and an object  $Y$  in  $\mathcal{D}'$ , we prove that the opcartesian lift of  $f$  at  $Y$  in  $\mathcal{D}'$  is given by  $\bar{f} \circ \eta_Z: Y \rightarrow LZ$ , where  $\bar{f}: Y \rightarrow Z$  is the opcartesian lift of  $f$  at  $Y$  with respect to  $F$  and  $\eta$  is the unit of the adjunction. Note that  $F\eta$  is an isomorphism because the adjunction is in  $\mathbf{CAT}/\mathcal{C}$ . For any morphism  $h: X \rightarrow FW$  with  $W$  in  $\mathcal{D}'$  and any lift  $k: Y \rightarrow W$  of  $f \circ h$ , there exists a unique morphism  $\bar{h}: Z \rightarrow W$  in  $\mathcal{D}$  such that  $F\bar{h} = h$  and  $\bar{f} \circ \bar{h} = k$ . Since  $W$  is in  $\mathcal{D}'$ , by the adjunction, there exists a unique morphism  $\tilde{h}: LZ \rightarrow W$  in  $\mathcal{D}'$  such that  $\eta_W \circ \tilde{h} = \bar{h}$ . It is easy to see that  $\tilde{h}$  is the unique morphism we want.  $\square$

**Remark 1.36.** Note that the opcartesian lifts in  $\mathcal{D}'$  constructed above do not necessarily coincide with those in  $\mathcal{D}$  even if the domain lies in  $\mathcal{D}'$ .  $\lrcorner$

**Proposition 1.37.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The functor  $|-|: \mathbf{CSh}(\mathcal{T}, \mathbf{J}) \rightarrow \mathbf{Set}$  at the terminal object  $1$  in  $\mathcal{T}$  is a bifibration.  $\lrcorner$

*Proof.* Note that  $(\mathbf{CSh}(\mathcal{T}, \mathbf{J}), |-|)$  is a full subcategory of  $([\mathcal{T}^{\text{op}}, \mathbf{Set}], (-)(1))$  in  $\mathbf{CAT}/\mathbf{Set}$  as in [Corollary 1.15](#).

To see that  $|-|$  is a fibration, we need to show that  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  is closed under cartesian lifts in  $[\mathcal{T}^{\text{op}}, \mathbf{Set}]$  as in [Proposition 1.35 \(1\)](#). This is straightforward from the explicit construction of cartesian lifts in [Lemma 1.34](#), as injections are closed under pullbacks.

It follows from [Proposition 1.35 \(2\)](#) that  $|-|$  is also an opfibration.  $\square$

It remains to show that the fibers of  $U$  have all coproducts. However, restricting to concrete sheaves forces the fibers to be preordered classes.

**Lemma 1.38.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site.  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$  over an object  $X$  in  $\mathbf{Set}$  with respect to the forgetful functor  $|-|: \mathbf{CSh}(\mathcal{C}, \mathbf{J}) \rightarrow \mathbf{Set}$  is a preordered class.  $\lrcorner$

*Proof.* Let  $A$  and  $B$  be objects in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$ , and take morphisms  $f, g: A \rightarrow B$  in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$ . For any  $U$  in  $\mathcal{C}$  and  $a \in A(U)$ , we have  $f_U(a) = g_U(a)$  because for each  $x: 1 \rightarrow U$  in  $\mathcal{C}$ , we have

$$x \cdot f_U(a) = f_1(x \cdot a) = x \cdot a = g_1(x \cdot a) = x \cdot g_U(a),$$

and  $B$  is separated for  $\text{Surj}$ . Thus,  $f = g$ .  $\square$

Finally, we show that the class of subobjects of an object in a presheaf category has all joins, which completes the proof of [Theorem 1.33](#).

**Lemma 1.39.** For a (possibly large) category  $\mathcal{C}$ , the class of subobjects of an object  $A$  in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  has all (possibly large) meets.  $\lrcorner$

*Proof.* Let  $\{A_i \hookrightarrow A\}_{i \in I}$  be a family of subobjects of  $A$ . The meet  $\bigwedge_{i \in I} A_i$  is given by the intersection  $\bigcap_{i \in I} A_i c$  in each component  $c \in \mathcal{C}$ . Indeed, these (large) intersections exist since each  $A c$  is a set. It is easy to see that  $\bigwedge_{i \in I} A_i$  is a presheaf, and the greatest lower bound of  $\{A_i \hookrightarrow A\}_{i \in I}$ .  $\square$

**Proposition 1.40.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. For every set  $X$ , the fiber  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$  over  $X$  with respect to the forgetful functor  $|-|: \mathbf{CSh}(\mathcal{C}, \mathbf{J}) \rightarrow \mathbf{Set}$  has all (possibly large) products.  $\lrcorner$

*Proof.* Any family of objects in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$  has a meet in the subobject preordered class of  $X^{|-|}$  in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  by [Lemma 1.39](#). This meet is a concrete sheaf over  $X$ , as separated presheaves are closed under subobjects and the sheaf condition for  $\mathbf{J}$  is closed under intersections by [Proposition 1.11](#). Since  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$  is a full subcategory of this preordered class by [Lemma 1.38](#), we conclude that  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})_X$  has all products.  $\square$

*Proof of Theorem 1.33.* It follows from the previous two propositions [Propositions 1.37](#) and [1.40](#) and the characterization of topological functors [Corollary 1.18](#).  $\square$

**Corollary 1.41.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is complete and cocomplete, and the forgetful functor  $|-|: \mathbf{CSh}(\mathcal{C}, \mathbf{J}) \rightarrow \mathbf{Set}$  preserves all limits and colimits.  $\lrcorner$

*Proof.* [Theorem 1.33](#) and [Proposition 1.20](#).  $\square$

**Corollary 1.42.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  has disjoint small coproducts.  $\lrcorner$

*Proof.* This is a direct consequence of [Corollary 1.41](#) and the fact that the only concrete sheaf over the empty set is the initial object, which is the empty presheaf.  $\square$

Since there is a morphism that underlies a bijection but is not an isomorphism in general, it follows that  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is not balanced, hence not a topos.

**Proposition 1.43 (Characterizations of initial sources and effectively epimorphic sinks).**

Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site.

- (1) For a source  $\{f_i: A \rightarrow A_i\}_{i \in I}$  in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is initial if and only if

$$|U| \xrightarrow{a} |A| \in A(U) \iff \forall i \in I \quad |U| \xrightarrow{a} |A| \xrightarrow{|f_i|} |A_i| \in A_i(U).$$

- (2) For a sink  $\{f_i: A_i \rightarrow A\}_{i \in I}$  in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is effectively epimorphic if and only if the underlying sink  $\{|f_i|: |A_i| \rightarrow |A|\}_{i \in I}$  is jointly surjective and the following condition holds:

$$\begin{array}{ccc} |U| & & |U_j| \xrightarrow{|f_j|} |U| \\ \downarrow a \in A(U) \iff & \exists \text{ J-covering fam. } \{ \iota_j: U_j \rightarrow U \}_{j \in J}, & \text{ s.t. } \begin{array}{ccc} a_j \downarrow & \circlearrowleft & \downarrow a \\ |A_{\alpha(j)}| & \xrightarrow{|f_{\alpha(j)}|} & |A| \end{array} \\ \downarrow & \exists \alpha: J \rightarrow I \in \mathbf{Set}, & \\ |A| & \exists a_j \in A_{\alpha(j)}(U_j) \quad (\forall j \in J) & \end{array}$$

(Note that the direction  $\Leftarrow$  is automatically satisfied, since if the right-hand side holds, then  $a_j \circ |f_{\alpha(j)}|$  is in  $A(U_j)$  for all  $j \in J$ , and thus  $a$  is in  $A(U)$  because  $A$  is a concrete sheaf.)  $\lrcorner$

**Remark 1.44.** When  $|f_i|$  and  $A_i$  are given, one can define a concrete sheaf structure on  $|A|$  by the right-hand sides in each condition above. When  $|f_i|$  are not jointly surjective, the right-hand side of the condition in (2) does not give a concrete sheaf, as constant functions at points outside the image of the sink do not live in the right-hand side.  $\lrcorner$

**Proposition 1.45 (Characterization of monomorphisms and epimorphisms).** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site.

- (1) A morphism  $f: A \rightarrow B$  in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is a monomorphism (resp. epimorphism) if and only if  $|f|$  is an injection (resp. surjection).  
(2) Strong epimorphisms and regular epimorphisms in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  coincide and are exactly the morphisms  $f: A \rightarrow B$  such that  $|f|$  is a surjection and the following condition holds:

$$\begin{array}{ccc} |U| & & |U_j| \xrightarrow{\iota_j} |U| \\ \downarrow b \in B(U) \iff & \exists \text{ J-covering fam. } \{ \iota_j: U_j \rightarrow U \}_{j \in J}, & \text{ s.t. } \begin{array}{ccc} a_j \downarrow & \circlearrowleft & \downarrow b \\ |A| & \xrightarrow{|f|} & |B| \end{array} \\ \downarrow & \exists a_j \in A(U_j) \quad (\forall j \in J) & \end{array}$$

- (3) Strong monomorphisms and regular monomorphisms in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  coincide and are exactly the morphisms  $f: A \rightarrow B$  such that  $|f|$  is an injection and the following condition holds:

$$|U| \xrightarrow{a} |A| \in A(U) \iff |U| \xrightarrow{a} |A| \xrightarrow{|f|} |B| \in B(U),$$

or equivalently,  $|f|$  is an injection and the concrete sheaf structure on  $A$  is inherited from that on  $B$  via  $f$ . In particular, strong subobjects are uniquely defined by their underlying subsets, that is, we have  $\mathbf{StrSub}_{\mathbf{CSh}(\mathcal{C}, \mathbf{J})}(-) \cong \mathbf{Sub}_{\mathbf{Set}}(|-|)$ .  $\lrcorner$

*Proof.* These characterizations follow from [Propositions 1.21](#) and [1.43](#). The last statement in the third item holds because  $| - |$  preserves pullbacks.  $\square$

**Remark’ 1.46.** In [DE06a, §4], the author develops a general theory of quasispaces on a site  $\mathcal{C}$  over a general base category, generalizing in a way that  $|P|$  is an object in the base category instead of a set. However, they do not correspond to what we call concrete sheaves in this note; a quasispaces on a concrete site  $\mathcal{C}$  (over  $\mathbf{Set}$ ) is a set  $|P|$  equipped with a family of subsets  $P(U) \subseteq \mathbf{Set}(U, |P|)$  for each  $U$  in  $\mathcal{C}$  with the same conditions derived from  $P$  being a concrete sheaf, except that  $P(1)$  is not the same as  $|P|$  in general.

Concrete sheaves correspond to what the author calls *strict quasispaces* (§5 in the paper), but they do not form a topological category over a category. The author provides a sufficient condition for the base category to ensure this is the case, and in particular, the category  $\mathbf{Set}$  satisfies this condition.

I did not include this generalization in this note, as it is not clear how it fits into the picture of the adjunction between concrete sheaves and topological spaces.  $\lrcorner$

**1.5. Quasitopos structure of the category of concrete sheaves.** Quasitoposes [Pen77] are a generalization of toposes in a way that there is only a classifier for strong subobjects. When you think about just sheaves on a large site, a sheaf can have a proper class of subobjects, so it is unlikely that this category has a subobject classifier. The category of sheaves on a large site is “too big” to be a topos. Nevertheless, concrete sheaves have only a set of strong subobjects by Proposition 1.45, so it is relatively reasonable to expect that the category of concrete sheaves has a classifier for them, which is indeed the case. We will give a direct proof of this, based on the theory of topological functors.

**Remark’ 1.47.** In [Dub79], Dubuc introduced the notion of a *concrete quasitopos*. The definition does not look like a specialization of the usual definition of a quasitopos, but the paper characterizes concrete quasitoposes as categories of quasispaces (concrete sheaves) on a concrete site with a standard topology and shows that they are quasitoposes in the usual sense. In [DE06a], it is shown what kind of conditions on a functor into a quasitopos make the domain a quasitopos. In the paper, the class of categories between Grothendieck quasitoposes and (elementary) quasitoposes, which they call *f-quasitoposes*, is introduced, and they show that a category of quasispaces is an f-quasitopos using their result. In this note, we do not follow this approach explicitly, but the core ideas are the same.  $\lrcorner$

**Definition 1.48.** A *strong-subobject classifier*<sup>9</sup> in a category  $\mathcal{C}$  is an object  $\Omega$  that represents the functor  $\mathbf{StrSub}(-): \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  sending an object  $A$  to the set of strong subobjects of  $A$  (i.e. the set of isomorphism classes of strong monomorphisms with codomain  $A$ ). The universal morphism  $\top: 1 \rightarrow \Omega$ , whose domain is automatically a terminal object, is called the *generic strong subobject*.  $\lrcorner$

**Definition 1.49 (Quasitopos).** A *quasitopos* is a category  $\mathcal{C}$  that has finite limits and colimits, is locally cartesian closed, and has a strong-subobject classifier.  $\lrcorner$

**Lemma 1.50.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is locally cartesian closed.  $\lrcorner$

*Proof.* By Corollary 1.31, it suffices to show that effectively epimorphic sinks in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  are stable under pullbacks. Take an effectively epimorphic sink  $\{f_i: A_i \rightarrow A\}_{i \in I}$  and a morphism  $g: B \rightarrow A$  in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$ , and let  $\{h_i: B_i \rightarrow B\}_{i \in I}$  be the sink obtained by pulling back the sink along  $g$ . We prove that  $\{h_i: B_i \rightarrow B\}_{i \in I}$  is effectively epimorphic using Proposition 1.43 (2). Suppose we have a plot  $b: |U| \rightarrow |B|$  in  $B(U)$  for some  $U$  in  $\mathcal{C}$ . The composite  $b \circ g$  is in  $A(U)$ , so by the condition in Proposition 1.43 (2), there exists a  $\mathbf{J}$ -covering family  $\{\iota_j: U_j \rightarrow U\}_{j \in J}$ , a function  $\alpha: J \rightarrow I$ , and plots

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<sup>9</sup>In the literature, a strong-subobject classifier is often called a *weak subobject classifier*, but we adopt the former terminology here.

$a_j \in A_{\alpha(j)}(U_j)$  such that the following outer square commutes:

$$\begin{array}{ccc}
 |U_j| & \xrightarrow{|\iota_j|} & |U| \\
 \downarrow & \circlearrowleft & \downarrow b \\
 |B_{\alpha(j)}| & \xrightarrow{|h_{\alpha(j)}|} & |B| \\
 \downarrow & \lrcorner & \downarrow g \\
 |A_{\alpha(j)}| & \xrightarrow{|f_{\alpha(j)}|} & |A|
 \end{array} \quad (j \in J).$$

Then, the induced function  $\langle a_j, |\iota_j| \circ b \rangle: |U_j| \rightarrow |B_{\alpha(j)}|$  is in  $B_{\alpha(j)}(U_j)$  for all  $j \in J$  because  $B_{\alpha(j)}$  is the pullback of  $f_{\alpha(j)}$  along  $g$ . This shows that the sink  $\{h_i: B_i \rightarrow B\}_{i \in I}$  is effectively epimorphic.  $\square$

**Lemma 1.51.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  has a strong-subobject classifier, given by  $\nabla\Omega$  where  $\Omega$  is the subobject classifier in  $\mathbf{Set}$ .  $\lrcorner$

*Proof.* We have the following bijection natural in  $A$  in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$ :

$$\text{StrSub}_{\mathbf{CSh}(\mathcal{C}, \mathbf{J})}(A) \xrightarrow[\text{Proposition 1.45}]{\cong} \text{Sub}_{\mathbf{Set}}(|-|)(|A|) \xrightarrow{\cong} \mathbf{Set}(|A|, \Omega) \xrightarrow[\text{Proposition 1.14}]{\cong} \mathbf{CSh}(\mathcal{C}, \mathbf{J})(A, \nabla\Omega). \quad \square$$

**Theorem 1.52.** Let  $(\mathcal{C}, \mathbf{J})$  be a concrete site. The category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is a quasitopos. It is also an small-extensive category<sup>10</sup>, that is, it has small coproducts that are disjoint and universal.  $\lrcorner$

*Proof.* The first statement follows from [Corollary 1.41](#) and [Lemmas 1.50](#) and [1.51](#).  $\square$

**Remark 1.53.** We note that  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  is not a topos in general. Indeed, every morphism in  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  that underlies a bijection is monic and epic at the same time by [Proposition 1.45](#). If the category were a topos, such a morphism would necessarily be an isomorphism, but this means that the topological functor  $|-|: \mathbf{CSh}(\mathcal{C}, \mathbf{J}) \rightarrow \mathbf{Set}$  would be contractible, so  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  would be equivalent to  $\mathbf{Set}$ .  $\lrcorner$

**1.6. Examples of concrete sheaves: subcategories of  $\mathbf{Top}$ .** We will present examples of concrete sheaves from the topological context, and by doing so, we will also set up the framework for the adjunction between concrete sheaves and topological spaces in the next section. The terminology of *prototypes* is just for this note.

**Definition 1.54 (Category of prototypes).** Let  $\mathcal{T}$  be a subcategory of the category of topological spaces  $\mathbf{Top}$  that is

- (1) closed under terminal object  $1 = \{*\}$ , and
- (2) contains all points (i.e., morphisms from 1).

From now on, we call such a category  $\mathcal{T}$  a *category of prototypes* (in  $\mathbf{Top}$ ) in this note, and the inclusion functor is denoted by  $I_{\mathcal{T}}: \mathcal{T} \rightarrow \mathbf{Top}$ .  $\lrcorner$

Note that we do not assume that  $\mathcal{T}$  is full in  $\mathbf{Top}$ . By the assumptions (1) and (2), the functor  $|-| = \mathcal{T}(1, -): \mathcal{T} \rightarrow \mathbf{Set}$  can be identified with the underlying set functor  $\mathcal{T} \rightarrow \mathbf{Set}$  that sends a topological space to its underlying set.

**Definition 1.55 (Site of prototypes).** Let  $\mathcal{T}$  be a category of prototypes. We fix a coverage  $\mathbf{J}$  on  $\mathcal{T}$ . A *site of prototypes* is a pair  $(\mathcal{T}, \mathbf{J})$  with a category of prototypes  $\mathcal{T}$  satisfying the following conditions:

- (1) every covering family  $\{\iota_i: U_i \rightarrow U\}_{i \in I}$  in  $\mathbf{J}$  is jointly surjective, that is, every point  $x: 1 \rightarrow U$  factors through some  $\iota_i: U_i \rightarrow U$  in the family, and
- (2) every presheaf of the form  $\mathbf{Top}(I_{\mathcal{T}}(-), X): \mathcal{T}^{\text{op}} \rightarrow \mathbf{Set}$  for a topological space  $X$  is a sheaf for the coverage  $\mathbf{J}$ .  $\lrcorner$

<sup>10</sup>In [\[Joh02a, A1.4\]](#), this is called an  $\infty$ -positive category. We adopt this terminology because it seems more common to call extensive categories rather than positive categories, and the adjective  $\infty$ - has been taken by  $\infty$ -categories.

**Remark 1.56.** By the first condition, a site of prototypes is a concrete site, and thus we can talk about concrete sheaves on it. The last condition is an analogue of the subcanonicity for Grothendieck topologies but relative to the inclusion  $I_{\mathcal{T}}: \mathcal{T} \rightarrow \mathbf{Top}$ , which is not necessary for what we discussed in this section, but it is important for the adjunction between concrete sheaves and topological spaces in the next section.  $\lrcorner$

**Example 1.57.** We give some examples of sites of prototypes.

- (1) If  $\mathcal{T}$  is a category of prototypes equipped with the trivial topology, then  $(\mathcal{T}, \text{Triv})$  is a site of prototypes. Every presheaf on  $\mathcal{T}$  is a sheaf for the trivial topology, so concrete sheaves on  $(\mathcal{T}, \text{Triv})$  are just concrete presheaves.
- (2) Suppose  $\mathcal{T}$  is a category of prototypes. By restricting the open cover topology on  $\mathbf{Top}$  to  $\mathcal{T}$ , we obtain a Grothendieck topology on  $\mathcal{T}$ , which we call the *open cover topology* and denote by  $\text{OpCov}$ . If  $\mathcal{T}$  is closed under open inclusions, the topology  $\text{OpCov}$  is generated by open covers in  $\mathcal{T}$ . The important examples of this type are the following:
  - (a) Let  $\mathbf{Mfd}$  be the category of smooth manifolds and smooth maps, which is a non-full subcategory of  $\mathbf{Top}$ . An equivalent (or more precisely, dense) subsite is given by the full subcategory consisting of open subsets of  $\mathbb{R}^n$  for some  $n \in \mathbb{N}$  with the (good) open cover topology. We will write this subcategory as  $\mathbf{Euc}$ . We can further restrict to just  $\mathbb{R}^n$  for  $n \in \mathbb{N}$  with the (good) open cover topology, which is denoted by  $\mathbf{CartSp}$ . Concrete sheaves on these equivalent sites are called *diffeological spaces* [ref](#). Sheaves on them are called *smooth sets*, and they form a topos as they are small sites. We could give examples of other flavors of manifolds, such as  $C^k$ -manifolds, complex manifolds, etc., by changing the class of morphisms, and they are also examples of sites of prototypes.
  - (b) The subcategory  $\mathbf{Cvx}$  of  $\mathbf{Top}$  consisting of convex subsets of  $\mathbb{R}^n$  for some  $n \in \mathbb{N}$  and smooth maps between them. Concrete sheaves on  $\mathbf{Cvx}$  are called *Chen spaces* [ref](#).
  - (c) The full subcategory  $\mathbf{CptHdf}$  of  $\mathbf{Top}$  consisting of compact Hausdorff spaces, and its concrete sheaves are called *quasi-topological spaces* [\[Spa63\]](#).

The condition [Definition 1.55 \(2\)](#) is satisfied because continuous maps can be glued along open covers when they agree on the overlaps and these subclasses of continuous maps (such as smooth maps) are closed under this process.

- (3) Let  $\mathbb{N}_{\infty}$  be the topological space obtained by the one-point compactification of the discrete space on the set of natural numbers  $\mathbb{N}$ . Morphisms from  $\mathbb{N}_{\infty}$  to a topological space  $X$  are in bijection with sequences  $(x_n)_n$  in  $X$  that converge to a point  $x_{\infty}$  in  $X$ . Consider the full subcategory  $\mathcal{N}_{\infty}$  of  $\mathbf{Top}$  consisting of  $\mathbb{N}_{\infty}$  and the terminal space. This is a category of prototypes. The canonical topology on  $\mathcal{N}_{\infty}$  is described in [\[Joh79\]](#). It is generated by the family  $\{1 \xrightarrow{\text{id}_1} 1\}$  on 1 and the families on  $\mathbb{N}_{\infty}$  that satisfy the following:
  - All points  $n: 1 \rightarrow \mathbb{N}_{\infty}$  for  $n \in \mathbb{N}$  are included in the family, and
  - For any infinite subset  $S \subseteq \mathbb{N}$ , there exists an infinite subset  $T \subseteq S$  such that the function  $\mathbb{N}_{\infty} \rightarrow \mathbb{N}_{\infty}$  enumerating numbers in  $T$  and sending  $\infty$  to  $\infty$  is included in the family.

[Definition 1.55 \(2\)](#) is satisfied essentially because for any sequence  $(x_n)_n$ , if every subsequence of it has a subsequence that converges to  $x_{\infty}$ , then the sequence itself converges to  $x_{\infty}$ . Note that this is not equivalent to the open cover topology, as the sieve generated by

$$n: 1 \rightarrow \mathbb{N}_{\infty} \quad (n \in \mathbb{N}), \quad \infty: 1 \rightarrow \mathbb{N}_{\infty}, \quad 2 \cdot -: \mathbb{N}_{\infty} \rightarrow \mathbb{N}_{\infty}, \quad 2 \cdot - + 1: \mathbb{N}_{\infty} \rightarrow \mathbb{N}_{\infty}$$

is covering in this topology but not in the open cover topology. Concrete sheaves on this site are equivalent to *subsequential spaces* (also known as *Kuratowski limit spaces*), and sheaves on this site are known as *(Johnstone's) topological topos* [\[Joh79\]](#).

- (4) The category of compact Hausdorff spaces  $\mathbf{CptHdf}$  can also be equipped with the coverage  $\mathbf{FinSurj}$  consisting of finite families of morphisms that are jointly surjective. Sheaves for this coverage are (equivalent to what are) called *condensed sets* [\[Sch19, Proposition 2.3\]](#), although there are some size issues to be dealt with in this case. Concrete sheaves for this coverage are also interesting, but I could not find a name for them in the literature (see [Section 3.2 \(4\)](#)).  $\lrcorner$

## 2. THE ADJUNCTION BETWEEN CONCRETE SHEAVES AND TOPOLOGICAL SPACES

In this section, we focus on concrete sheaves on a site of prototypes and construct an adjunction between the category of concrete sheaves and the category of topological spaces.

**2.1. Constructing the adjunction.** We first construct a “**nerve-and-realization**”<sup>11</sup> adjunction between  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  and  $\mathbf{Top}$ . Since  $\mathcal{T}$  is not necessarily small, we need to be careful again, although nothing serious happens here.

**Proposition 2.1.** Let  $\mathcal{T}$  be a category of prototypes. There exists an adjunction

$$[\mathcal{T}^{\text{op}}, \mathbf{Set}] \begin{array}{c} \xrightarrow{K'_{\mathcal{T}}} \\ \xleftarrow{\perp} \\ \xrightarrow{I_{\mathcal{T}}^*} \end{array} \mathbf{Top} ,$$

where  $I_{\mathcal{T}}^*$  is the nerve functor with respect to the inclusion  $I_{\mathcal{T}}$ , and  $K'_{\mathcal{T}}$  is its left Kan extension along the Yoneda embedding.

This is also an adjunction in  $\mathbf{CAT}/\mathbf{Set}$ , that is,  $K'_{\mathcal{T}}$  and  $I_{\mathcal{T}}^*$  are compatible with the underlying set functors.  $\lrcorner$

*Proof.* What departs from the ordinary proof is to show that the left Kan extension  $K_{\mathcal{T}}$  exists, which follows from the fact that  $\mathbf{Top}$  is topological over  $\mathbf{Set}$ , and colimits therein are constructed on underlying sets. For a presheaf  $P$  in  $[\mathcal{T}^{\text{op}}, \mathbf{Set}]$ , the colimit of the diagram  $|-| \circ I_{\mathcal{T}}: \mathcal{T} \rightarrow \mathbf{Top}$  weighted by  $P$  exists because  $|-| \circ I_{\mathcal{T}} \cong \mathcal{T}(1, -)$  and we have the following calculation:

$$\begin{aligned} \text{colim}^P(|-| \circ I_{\mathcal{T}}) &\cong \text{colim}^P \mathcal{T}(1, -) \\ &\cong \text{colim}^{\mathcal{T}(1, -)} P && \text{(the exchange rule)} \\ &\cong P1. && \text{(the co-Yoneda lemma)} \end{aligned}$$

Therefore, the colimit  $\text{colim}^P I_{\mathcal{T}}$  underlies the set  $P1$ , and the topology is the final topology with respect to the family of maps  $\{\gamma_U(p): U \rightarrow P1 \mid U \in \mathcal{T}, p \in PU\}$ , where  $\gamma_U$  is the canonical map (1.1). From this description, it is easy to see that  $K'_{\mathcal{T}}$  is compatible with the underlying set functors.  $I_{\mathcal{T}}^*$  is also compatible with the underlying set functors because

$$(I_{\mathcal{T}}^*(S))(1) = \mathbf{Top}(I_{\mathcal{T}}(1), S) \cong \mathbf{Top}(1, S) \cong |S|. \quad \square$$

**Theorem 2.2.** Let  $(\mathcal{T}, \mathbf{J})$  be a site of prototypes. The adjunction in Proposition 2.1 restricts to an adjunction

$$\mathbf{CSh}(\mathcal{T}, \mathbf{J}) \begin{array}{c} \xrightarrow{K_{\mathcal{T}}} \\ \xleftarrow{\perp} \\ \xrightarrow{N_{\mathcal{T}}} \end{array} \mathbf{Top} .$$

This is also an adjunction in  $\mathbf{CAT}/\mathbf{Set}$ .  $\lrcorner$

*Proof.* We show that the right adjoint  $I_{\mathcal{T}}^*$  restricts to  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$ . Let  $S$  be a topological space. By the last assumption in Definition 1.55,  $I_{\mathcal{T}}^*(S)$  is a  $\mathbf{J}$ -sheaf. Moreover, it is concrete because for every  $U$  in  $\mathcal{T}$ ,

$$\mathbf{Top}(U, S) \rightarrow \mathbf{Top}(1, S)^{|U|} \cong |S|^{|U|}; \quad f \mapsto (x \mapsto f \circ x)$$

is injective. Thus,  $I_{\mathcal{T}}^*(S)$  is a concrete  $\mathbf{J}$ -sheaf.  $\square$

**Example 2.3.** For  $(\mathbf{Mfd}, \mathbf{OpCov})$ , the adjunction in Theorem 2.2 gives an adjunction between diffeological spaces and topological spaces, and the topology on  $K_{\mathbf{Mfd}}(P)$  for a diffeological space  $P$  is called the *D-topology* [Igl13].  $\lrcorner$

**Corollary 2.4.** Let  $(\mathcal{T}, \mathbf{J})$  be a site of prototypes. The adjunction in Proposition 2.1 is the composite of

$$[\mathcal{T}^{\text{op}}, \mathbf{Set}] \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{\perp} \end{array} \mathbf{CSh}(\mathcal{T}, \mathbf{J}) \begin{array}{c} \xrightarrow{K_{\mathcal{T}}} \\ \xleftarrow{\perp} \\ \xrightarrow{N_{\mathcal{T}}} \end{array} \mathbf{Top} ,$$

where  $L$  is the left adjoint of  $\mathbf{CSh}(\mathcal{T}, \mathbf{J}) \hookrightarrow [\mathcal{T}^{\text{op}}, \mathbf{Set}]$  as in Proposition 1.12.

<sup>11</sup>The same concept is called a *universal adjunction* in a Japanese online textbook [alg] on category theory, but more standard terminology seems to be lacking except for “the nerve and realization.”

In particular, the essential image of  $K'_\mathcal{T}$  is the same as that of  $K_\mathcal{T}$ , and the comonad induced by the adjunction in [Proposition 2.1](#) on **Top** is the same as that induced by the adjunction in [Theorem 2.2](#).  $\lrcorner$

*Proof.* Since the inclusion  $\mathbf{CSh}(\mathcal{T}, \mathbf{J}) \hookrightarrow [\mathcal{T}^{\text{op}}, \mathbf{Set}]$  commutes with the right adjoints, the left adjoints also commute.  $\square$

**Proposition 2.5.** Let  $(\mathcal{T}, \mathbf{J})$  be a site of prototypes. The yoneda embedding  $\mathfrak{y}: \mathcal{T} \rightarrow [\mathcal{T}^{\text{op}}, \mathbf{Set}]$  factors through  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  if and only if it is subcanonical, in which case it gives an embedding  $\mathcal{T} \rightarrow \mathbf{CSh}(\mathcal{T}, \mathbf{J})$ .  $\lrcorner$

*Proof.* For every  $U$  in  $\mathcal{T}$ , the presheaf  $\mathcal{T}(-, U)$  is naturally seen as a subfunctor of  $|U|^{|\cdot|}$ . Thus, if the Yoneda embedding factors through  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  if and only if  $\mathcal{T}(-, U)$  is a  $\mathbf{J}$ -sheaf for every  $U$  in  $\mathcal{T}$ , which is exactly the subcanonicity condition.  $\square$

**Example 2.6.** The sites of prototypes in [Example 1.57](#) are all subcanonical, so the Yoneda embedding factors through  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  in all those examples. For instance, this gives a fully faithful embedding  $\mathbf{Mfd} \rightarrow \mathbf{CSh}(\mathbf{Mfd}, \mathbf{OpCov})$ , which is the usual inclusion of smooth manifolds into diffeological spaces.  $\lrcorner$

**Definition 2.7.** The idempotent comonad on **Top** induced by the adjunction in [Theorem 2.2](#) is written as  $k_\mathcal{T}: \mathbf{Top} \rightarrow \mathbf{Top}$ . The underlying set of  $k_\mathcal{T}S$  for a space  $S$  is identified with  $|S|$  by the last sentence in [Proposition 2.1](#), and its topology is called the  $\mathcal{T}$ -generated topology on  $|S|$ . Explicitly,

$$k_\mathcal{T}S = \text{colim}^{\mathbf{Top}(I_\mathcal{T}(-), S)} I_\mathcal{T}. \quad \lrcorner$$

**Remark 2.8.** By the concrete description of colimits using final lifts for  $\mathbf{Top} \rightarrow \mathbf{Set}$ , the  $\mathcal{T}$ -generated topology on  $|S|$  is the finest topology such that all continuous functions  $U \rightarrow S$  from some  $U$  in  $\mathcal{T}$  (with respect to the original topology on  $S$ ) are continuous. In terms of open sets, it means that a subset  $V$  of  $S$  is an open set in  $k_\mathcal{T}S$  if and only if when  $f^{-1}(V)$  is open in  $U$  for every  $U$  in  $\mathcal{T}$  and every continuous map  $f: U \rightarrow S$ .  $\lrcorner$

**2.2. The idempotentness of the adjunction.** We recall several equivalent characterizations of an [idempotent adjunction](#).<sup>12</sup>

**Theorem 2.9.** For an adjunction  $F: \mathcal{A} \rightleftarrows \mathcal{B}: G$  with unit  $\eta$  and counit  $\varepsilon$ , the following are equivalent:

- (1)  $F\eta$  is an isomorphism.
- (2)  $\eta G$  is an isomorphism.
- (3)  $G\varepsilon$  is an isomorphism.
- (4)  $\varepsilon F$  is an isomorphism.
- (5) The induced monad  $GF$  is idempotent, i.e., the multiplication  $G\varepsilon F: GF GF \rightarrow GF$  is an isomorphism.
- (6) The induced comonad  $FG$  is idempotent, i.e., the comultiplication  $F\eta G: FG \rightarrow FG FG$  is an isomorphism.
- (7) An object  $a$  in  $\mathcal{A}$  is in the essential image of  $G$  if and only if the unit component  $\eta_a: a \rightarrow GFa$  is an isomorphism.
- (8) An object  $b$  in  $\mathcal{B}$  is in the essential image of  $F$  if and only if the counit component  $\varepsilon_b: FGb \rightarrow b$  is an isomorphism.  $\lrcorner$

**Definition 2.10.** An adjunction satisfying the equivalent conditions above is called an *idempotent adjunction*.  $\lrcorner$

**Theorem 2.11.** Let  $\mathcal{T}$  be a category of prototypes. The adjunction in [Theorem 2.2](#) is idempotent.  $\lrcorner$

I found the following proof on MathOverflow [[Aiv25](#)].

<sup>12</sup>There are more than those; we do not include the conditions using the equations of natural transformations, such as  $GF\eta = \eta GF$ .



- (b) it is left orthogonal to every component of the counit  $\varepsilon_{b'} : kb' \rightarrow b'$ ,
- (c) the function  $\mathcal{B}(b', b) \rightarrow \mathcal{B}_k(b', b)$  induced by the cofree functor is bijective.  $\lrcorner$

**2.3.  $\mathcal{T}$ -generated spaces and  $\mathcal{T}$ -continuous maps.** We are now describing the fixpoints of the adjunction in [Theorem 2.2](#).

**Definition 2.14.** Let  $\mathcal{T}$  be a category of prototypes.  $\mathcal{T}$ -*generated spaces* are objects in the essential image of the left adjoint  $K_{\mathcal{T}}$  in [Theorem 2.2](#). We denote the full subcategory of  $\mathbf{Top}$  consisting of  $\mathcal{T}$ -generated spaces by  $\mathbf{Top}_{\mathcal{T}}$ .  $\lrcorner$

**Remark 2.15.** By [Lemma 2.13 \(2\)](#), being  $\mathcal{T}$ -generated is a property on a topological space.  $\lrcorner$

[Theorem 2.12](#) implies that the adjunction factors as the following reflection and coreflection:

$$\mathbf{CPSH}(\mathcal{T}) \begin{array}{c} \xrightarrow{K_{\mathcal{T}}} \\ \xleftarrow{L_1} \mathbf{CSh}(\mathcal{T}, \mathbf{J}) \xrightarrow{K_{\mathcal{T}}} \\ \xleftarrow{\perp} \mathbf{Top}_{\mathcal{T}} \xleftarrow{k_{\mathcal{T}}} \mathbf{Top} \\ \xleftarrow{N_{\mathcal{T}}} \end{array}$$

**Remark 2.16.** The fix-point category does not change when we take different topologies on the category  $\mathcal{T}$ , as the comonads induced on  $\mathbf{Top}$  are the same ([Corollary 2.4](#)). Therefore, it suffices to discuss the most extreme case where  $\mathbf{J} = \mathbf{Triv}$  and  $\mathbf{CSh}(\mathcal{T}, \mathbf{J}) = \mathbf{CPSH}(\mathcal{T})$ .  $\lrcorner$

**Lemma 2.17.** Let  $\mathcal{T}$  be a category of prototypes. Every space in  $\mathcal{T}$  is  $\mathcal{T}$ -generated.  $\lrcorner$

*Proof.* Since  $K'_{\mathcal{T}}$  is the pointwise left Kan extension of the inclusion  $I_{\mathcal{T}}$  along  $\mathfrak{z}$ , which is fully faithful, we have that  $K'_{\mathcal{T}}\mathfrak{z} \cong I_{\mathcal{T}}$ , and hence  $K_{\mathcal{T}}(\mathfrak{z}_U) \cong U \in \mathbf{Top}_{\mathcal{T}}$  for each  $U$  in  $\mathcal{T}$ .  $\square$

**Definition 2.18.** Let  $\mathcal{T}$  be a category of prototypes. A  $\mathcal{T}$ -*continuous map* between topological spaces  $S$  and  $S'$  is a continuous map from  $k_{\mathcal{T}}S$  to  $S'$ . We denote the category of topological spaces and  $\mathcal{T}$ -continuous maps, or equivalently, the coKleisli category of the comonad  $k_{\mathcal{T}}$  on  $\mathbf{Top}$ , by  $\mathcal{T}\text{-Cont}$ .  $\lrcorner$

**Remark 2.19.** As pointed out earlier,  $\mathcal{T}\text{-Cont}$  is also the fix-point category of the adjunction in [Theorem 2.2](#). It is also topological over  $\mathbf{Set}$ , so being a  $\mathcal{T}$ -continuous map is a property on the underlying function. Continuous maps are  $\mathcal{T}$ -continuous.  $\lrcorner$

**Remark 2.20.** By the concrete description of the  $\mathcal{T}$ -generated topology, a function  $f : |S| \rightarrow |S'|$  is  $\mathcal{T}$ -continuous if and only if for every continuous map  $g : U \rightarrow S$  from some  $U$  in  $\mathcal{T}$ , the composite  $|g| \circ f : |U| \rightarrow |S'|$  is continuous with respect to the original topology on  $S'$ .  $\lrcorner$

**Example 2.21.** There are many examples of  $\mathcal{T}$ -generated spaces, and they are often useful for doing homotopy theory in  $\mathbf{Top}$ .

- (1)  $\mathbf{CptHdf}$ -generated spaces are called *compactly generated spaces* [[Ste67](#); [Wyl73](#)].
- (2)  $\mathbf{Euc}$ -generated spaces are called  *$\Delta$ -generated spaces* [[Dug03](#)] (or *numerically generated spaces* [[SYH18](#)]). These are also the spaces generated by all topological simplices  $\Delta_n \in \mathbf{Top}$ , which is where the name comes from.
- (3)  $\mathcal{N}_{\infty} = \{1, \mathbb{N}_{\infty}\}$ -generated spaces are called *sequential spaces*.  $\mathcal{N}_{\infty}$ -continuous maps are called *sequentially continuous maps*; explicitly, a function is sequentially continuous if it preserves limits of convergent sequences (see the following proposition).
- (4)  $\{1\}$ -generated spaces are just discrete spaces, as colimits of discrete spaces are discrete.
- (5) It is stated in [[ELS04](#)] that  $\{1, \mathbb{S}\}$ -generated spaces, where  $\mathbb{S}$  is the Sierpiński space, are exactly *Alexandroff spaces*, that is, spaces whose topology is determined by the specialization preorder. More examples are in Section 3 of that paper.  $\lrcorner$

**Proposition 2.22.** Let  $\mathcal{T}$  be a category of prototypes. The following are equivalent for a space  $S$  in  $\mathbf{Top}$ :

- (1)  $S$  is  $\mathcal{T}$ -generated,
- (2) the canonical continuous map  $k_{\mathcal{T}}S \rightarrow S$  is a homeomorphism,

- (3) every  $\mathcal{T}$ -continuous map into  $S$  is continuous,
- (4)  $S$  is a colimit of a (possibly large) diagram in  $\mathcal{T}$  that exists in  $\mathbf{Top}$ . ┘

*Proof.* The canonical map  $k_{\mathcal{T}}S \rightarrow S$  is the counit component of the adjunction in [Theorem 2.2](#), so (1) and (2) are equivalent by [Lemma 2.13 \(2\)](#). Again by [Lemma 2.13 \(2\)](#), (3) is equivalent to the above two conditions. We also see that (2) implies (4) because  $k_{\mathcal{T}}S$  is a colimit of a diagram in  $\mathcal{T}$ , and (4) implies (2) because  $\mathbf{Top}_{\mathcal{T}}$  is closed under colimits in  $\mathbf{Top}$  as a coreflective subcategory. □

**Proposition 2.23.** Let  $\mathcal{T}$  be a category of prototypes.

- (1) The category  $\mathbf{Top}_{\mathcal{T}}$  is complete and cocomplete, where colimits coincide with those in  $\mathbf{Top}$ , and limits are inherited from  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$ .
- (2) The category  $\mathbf{Top}_{\mathcal{T}}$  is closed under quotient: if  $S$  is a  $\mathcal{T}$ -generated space and there is a surjective final map  $S \rightarrow Q$  in  $\mathbf{Top}$ , then  $Q$  is also  $\mathcal{T}$ -generated. ┘

*Proof.*

- (1) It is both a reflective subcategory of  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$  and a coreflective subcategory of  $\mathbf{Top}$ .
- (2) Let  $N_{\mathcal{T}}(S)$  be the concrete presheaf corresponding to  $S$  and let  $N_{\mathcal{T}}(S) \rightarrow Q'$  be the final lift of the surjection  $|S| \rightarrow |Q|$  along the forgetful functor  $|-|: \mathbf{CSh}(\mathcal{T}) \rightarrow \mathbf{Set}$ . Since the left adjoint  $K_{\mathcal{T}}$  preserves final surjections, this is sent to a final surjection  $S \rightarrow K_{\mathcal{T}}(Q')$ , and the codomain can be identified with  $Q$ . □

**Remark 2.24.** It is also known to be true that

- (1) for any productive category  $\mathcal{T}$  of prototypes,  $\mathbf{Top}_{\mathcal{T}}$  is exponentiable in  $\mathbf{Top}$ ,
- (2) for any small category  $\mathcal{T}$  of prototypes,  $\mathbf{Top}_{\mathcal{T}}$  is a locally presentable category.

These results make these categories of  $\mathcal{T}$ -generated spaces deserving to be called convenient categories of spaces, but we will not reproduce the proofs here. The first result dates back to [\[Day72\]](#), but a more direct proof using  $\mathcal{T}$ -continuous maps is given in [\[ELS04, Section 3\]](#). The second result is shown in [\[FR07\]](#) using syntactic presentations of fiberwise small topological categories called *relational universal strict Horn theories*, but more direct proof is suggested on MathOverflow [\[Tim15\]](#). Since we know that  $\mathbf{Top}_{\mathcal{T}}$  is a reflective subcategory of  $\mathbf{CSh}(\mathcal{T}, \mathbf{J})$ , naive ideas would be to show that the reflector preserves finite products (resp. the inclusion is accessible), but it seems nontrivial to show these properties directly. ┘

### 3. MISCELLANEOUS TOPICS

**3.1. Two related concepts in topos theory: biseparated presheaves and local toposes.** Concrete sheaves are an example of a more general concept of biseparated presheaves.

**Definition 3.1.** Let  $\mathcal{T}$  be a category of prototypes. There is a coverage consisting of the families of morphisms  $\{x: 1 \rightarrow U\}$  for each  $U$  in  $\mathcal{T}$ , which we call the *surjective coverage*. ┘

The surjective topology is too fine for working with sheaves because the sheaves for this topology are just sets, but it is useful to describe concrete sheaves as biseparated presheaves for the bisite  $(\mathcal{T}, \mathbf{J}, \mathbf{Surj})$ .

**Definition 3.2 (Bisites and biseparated presheaf [\[Joh02b, C2.2\]](#)).** A *bisite* is a triple  $(\mathcal{C}, \mathbf{J}, \mathbf{K})$  where  $\mathcal{C}$  is a category and  $\mathbf{J}$  and  $\mathbf{K}$  are coverages on  $\mathcal{C}$  such that  $\mathbf{J}$  is coarser than  $\mathbf{K}$ . A presheaf  $P$  on  $\mathcal{C}$  is *biseparated with respect to  $(\mathbf{J}, \mathbf{K})$  (or  $(\mathbf{J}, \mathbf{K})$ -biseparated)* if it is separated for  $\mathbf{K}$  and a sheaf for  $\mathbf{J}$  (see [Definition 1.7](#)). ┘

The condition in the definition of sites of prototypes can now be rephrased as the coverage being coarser than  $\mathbf{Surj}$ . The points coverage  $\mathbf{Surj}$  is also obtained by restricting the surjective topology on  $\mathbf{Top}$  to  $\mathcal{T}$ .

**Remark' 3.3.** There are two possible ways to generalize the surjective topology on a more general category  $\mathcal{C}$ . One is to take the coverage consisting of families  $\{x: 1 \rightarrow U\}$  for each  $U$  in  $\mathcal{C}$ , and the other is to take the coverage consisting of families  $\{f_i: U_i \rightarrow U\}$  that underlie jointly surjective families

in **Set**. Both of them seem to require some additional assumptions on  $\mathcal{C}$  to be well-defined, and they do not coincide in general.  $\lrcorner$

Let us show what the separatedness condition for **Surj** means explicitly.

**Proposition 3.4.** Let  $\mathcal{T}$  be a category of prototypes. A presheaf  $P$  on  $\mathcal{T}$  is **Surj**-separated if and only if it is isomorphic to a concrete presheaf. (See [Remark 1.5](#) for the explicit description of the second condition.)  $\lrcorner$

*Proof.* For a **Surj**-covering family  $\{x: 1 \rightarrow U\}_{x \in |U|}$ , there are only two cases for the commutative square in (1.2): the first case is when  $i = j$  and  $k = \ell$ , and the second case is  $V$  is the empty space if it exists in  $\mathcal{T}$ . In the first case, the condition  $k \cdot p_i = \ell \cdot p_j$  is trivial. If  $\emptyset$  is in  $\mathcal{T}$ , the empty family on  $\emptyset$  is a **Surj**-covering family, so the separatedness condition implies that  $P\emptyset$  has at most one element, and thus the condition  $k \cdot p_i = \ell \cdot p_j$  is also trivial in the second case. In conclusion, the codomain set in (1.2) is just  $\prod_{x \in |U|} PU$ , which is  $P1^{|U|}$  for a covering family  $\{x: 1 \rightarrow U\}_{x \in |U|}$ .  $\square$

**Corollary 3.5.** Let  $(\mathcal{T}, \mathbf{J})$  be a site of prototypes. We have the following equivalence of categories:

$$\mathbf{CPSH}(\mathcal{T}) \simeq \mathbf{Sep}(\mathcal{T}, \mathbf{Surj}), \quad \mathbf{CSh}(\mathcal{T}, \mathbf{J}) \simeq \mathbf{BiSep}(\mathcal{T}, \mathbf{J}, \mathbf{Surj}),$$

where  $\mathbf{Sep}(\mathcal{T}, \mathbf{Surj})$  and  $\mathbf{BiSep}(\mathcal{T}, \mathbf{J}, \mathbf{Surj})$  are the full subcategories of  $[\mathcal{T}, \mathbf{Set}]$  consisting of **Surj**-separated presheaves and  $(\mathbf{J}, \mathbf{Surj})$ -biseparated presheaves, respectively.  $\lrcorner$

Finally, we briefly explain the relationship between concrete sheaves and local toposes.

**Definition 3.6.** Let  $\mathcal{C}$  be a small category with a terminal object 1. A site  $(\mathcal{C}, \mathbf{J})$  is said to be **local** if the maximal sieve on 1 is the only covering sieve on 1.  $\lrcorner$

Local sites are the site characterization of local toposes [[Joh02b](#), C3.6].

**Proposition 3.7.** For a small site  $(\mathcal{C}, \mathbf{J})$  on a category  $\mathcal{C}$  with a terminal object 1, the following are equivalent.

- (1)  $(\mathcal{C}, \mathbf{J})$  is concrete,
- (2)  $\mathcal{C}(1, -)$  is faithful and  $(\mathcal{C}, \mathbf{J})$  is local.  $\lrcorner$

*Proof.* When  $(\mathcal{C}, \mathbf{J})$  is concrete, a covering sieve on 1 should contain a morphism  $!_U: U \rightarrow 1$  with  $|U|$  nonempty. Then, the identity on 1, obtained by precomposing  $x \in |U| = \mathcal{C}(1, U)$  with  $!_U$ , is in this sieve. Conversely, suppose that  $(\mathcal{C}, \mathbf{J})$  is local and  $|-| = \mathcal{C}(1, -)$  is faithful. Take any  $\mathbf{J}$ -covering family  $\{\iota_i: U_i \rightarrow U\}_{i \in \mathbf{J}}$  and any morphism  $x: 1 \rightarrow U$ . By the stability, there is a  $\mathbf{J}$ -covering family  $\{\kappa_j: V_j \rightarrow 1\}_{j \in \mathbf{J}}$  over 1 such that  $\kappa_j \circ x$  factors through some  $\iota_i$  for every  $j$ . However, by assumption,  $\text{id}_1$  factors through some  $\kappa_j$ . Combining these two, we see that  $x$  itself factors through some  $\iota_i$ . This means that  $\{|\iota_i|\}_i$  is jointly surjective.  $\square$

**Corollary 3.8.** For a small concrete site  $(\mathcal{C}, \mathbf{J})$ , the category  $\mathbf{Sh}(\mathcal{C}, \mathbf{J})$  of sheaves on  $\mathcal{C}$  is a local topos (over **Set**).  $\lrcorner$

This means that the topos of sheaves on  $(\mathcal{C}, \mathbf{J})$  admits the following adjoint triple with  $\Delta$  and  $\Gamma$  being fully faithful:

$$\mathbf{Sh}(\mathcal{C}, \mathbf{J}) \begin{array}{c} \xleftarrow{\Delta} \\ \perp \\ \xrightarrow{|-|} \mathbf{Set} \\ \perp \\ \xleftarrow{\Gamma} \end{array} .$$

**Proposition 3.9.** For a small concrete site  $(\mathcal{C}, \mathbf{J})$ , the category  $\mathbf{CSh}(\mathcal{C}, \mathbf{J})$  of concrete sheaves on  $\mathcal{C}$  is equivalent to the subcategory of  $\mathbf{Sh}(\mathcal{C}, \mathbf{J})$  consisting of objects  $P$  that is separated with respect to sheaf morphisms that underlie bijections on the underlying sets, i.e., for every morphism  $f: Q \rightarrow R$  in  $\mathbf{Sh}(\mathcal{C}, \mathbf{J})$  such that  $|f|$  is a bijection, there exists at most one morphism  $g: R \rightarrow P$  for each morphism

$h: Q \rightarrow P$  such that the following diagram commutes.

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ f \downarrow & \nearrow g & \\ R & & \end{array}$$

□

*Proof.* Note that the latter condition is invariant up to isomorphism. Any concrete sheaf satisfies this condition because morphisms in  $\mathbf{CSh}(\mathcal{C}, \mathcal{J})$  are functions between the underlying sets with some additional properties, so there is only one candidate for  $g$ , which may or may not give the desired commutative diagram in  $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$ .

Conversely, if  $P$  satisfies the condition, then we take  $f$  to be the following morphism in  $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$ :

$$L([\mathfrak{J}_x]_x): L\left(\bigsqcup_{x \in U} \mathfrak{J}_1\right) \rightarrow L\mathfrak{J}_U,$$

where  $L$  is the reflector from  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  to  $\mathbf{CSh}(\mathcal{C}, \mathcal{J})$ . Its underlying function is a bijection since  $[\mathfrak{J}_x]_x$  has the underlying bijection, and the separatedness condition for this implies the condition in [Remark 1.5](#), so  $P$  is isomorphic to a concrete sheaf. □

**3.2. Future directions.** Here is the list that I think are interesting future directions to explore.

- (1) Morphisms between sites, possibly compatible with the inclusion functor into  $\mathbf{Top}$ , induce geometric morphisms between the corresponding categories of concrete sheaves.
- (2) To give the adjunction in [Theorem 2.2](#) as topological adjunction. The perspective of topological categories as enriched categories might offer a clearer viewpoint.
- (3) **Pseudotopological spaces** or *Choquet spaces* [[Joh02a](#), A2.6] are topological structures similar in spirit of concrete sheaves but not exactly an instance of them. The issue is that filters in a topological space cannot be identified with morphisms from a fixed object in  $\mathbf{Top}$ , but I suspect if one take nets instead of filters, then we can present the equivalent category of pseudotopological spaces as concrete sheaves on a site of prototypes ([Section 3.2 \(3\)](#)). Here is a sketch of the idea.

Let  $A$  be a directed poset, i.e., a poset where every finite subset has an upper bound. We can impose a topology on the set  $A_\infty := A \cup \{\infty\}$  by the basis consisting of  $\{a\}$  for each  $a \in A$  and  $A_{\geq a} \cup \{\infty\}$  for each  $a \in A$ , where  $A_{\geq a} := \{b \in A \mid b \geq a\}$ . This is a generalization of the one-point compactification of the natural numbers. A continuous map from  $A_\infty$  to a space  $X$  is identified with a net in  $X$  indexed by  $A$  together with its limit. Let  $\mathbf{Dir}$  be the full subcategory of  $\mathbf{Top}$  consisting of spaces  $A_\infty$  for directed posets  $A$ . Define a coverage on it by declaring a family of morphisms to be covering if and only if it contains every point  $1 = \emptyset_\infty \rightarrow A_\infty$  and for any unbounded subset  $B \subseteq A$ , it contains a subset  $C \subseteq B$  such that the continuous map  $C_\infty \rightarrow A_\infty$  induced by the inclusion  $C \hookrightarrow A$  is in the family.

Every topological space is  $\mathbf{Dir}$ -generated as net convergence is sufficient to determine the topology. For each topological space  $X$ , the presheaf  $\mathbf{Top}(i_{\mathbf{Dir}}(-), X)$  is a concrete sheaf with respect to this site structure because if every subnet of a net has a subnet converging to a point  $x$ , then the original net also converges to  $x$ , so this is a site of prototypes.

We have the following questions:

- What can we say about this topology? Is this the canonical topology? (Seemingly not.)
  - The concrete sheaves with respect to this site give another “convergence structures”. How are they related to pseudotopological spaces or other generalized convergence structures? (It should be equivalent as net convergence is “equivalent” to filter convergence.)
- (4) In condensed mathematics, some tricks are used to avoid set-theoretic issues caused by large sites. How much can we develop the parallel theory in the category of concrete sheaves for the finite surjective coverage on the site of compact Hausdorff spaces or profinite spaces?

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*Email address:* [hnasu@dal.ca](mailto:hnasu@dal.ca)