On the decomposition of a strong epimorphism into regular epimorphisms

Hayato Nasu (Dalhousie Univ.)

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jww. Yuto Kawase (Kyoto Univ.)

0. Introduction

1. The decomposition number

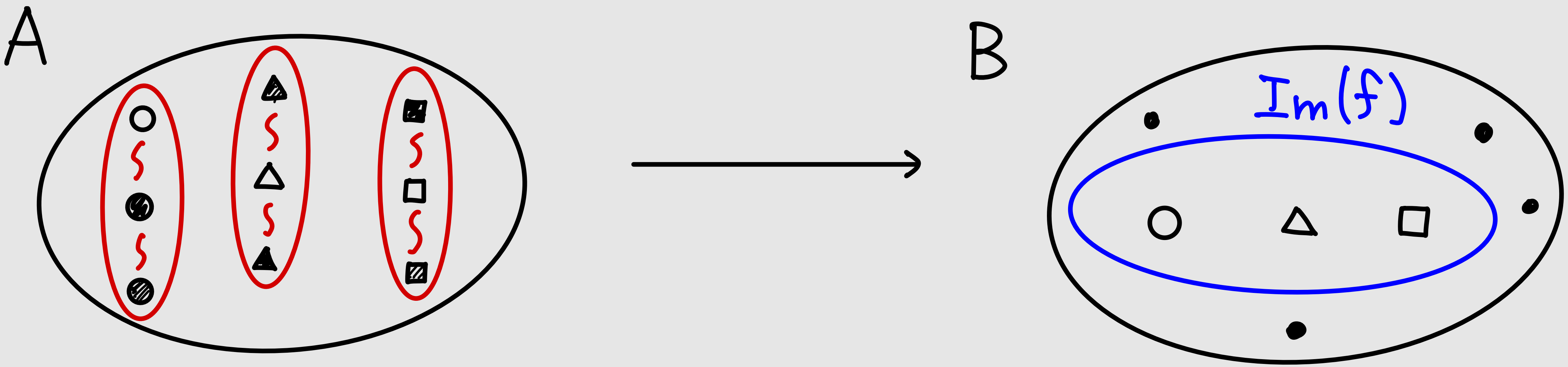
2. Finding the decomposition number
   via generalized algebraic theory

Yuto Kawase and Hayato Nasu,
"On the decomposition of a strong epimorphism into regular epimorphisms," arXiv:2604.05744.

The fundamental theorem of homomorphisms

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \quad \rightsquigarrow \quad A/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$$

Identifying two elements that go to the same element gives you the image!



- For groups $\varphi : G \to H \rightsquigarrow G/\underline{\mathrm{Ker}(\varphi)} \cong \mathrm{Im}(\varphi)$

  ↑ normal subgroup

- For rings $\varphi : R \to S \rightsquigarrow R/\underline{\mathrm{Ker}(\varphi)} \cong \mathrm{Im}(\varphi)$

  ↑ ideal

- For modules $\varphi : M \to N \rightsquigarrow M/\underline{\mathrm{Ker}(\varphi)} \cong \mathrm{Im}(\varphi)$

  ↑ submodule

- For sets $\varphi : X \to Y \rightsquigarrow X/\underline{\sim_\varphi} \cong \mathrm{Im}(\varphi)$

  ↑ equivalence relation

What about categories?

Counter example :

$$\mathrm{Im}(\Phi) = \boxed{\mathcal{D}}$$

$$\boxed{\mathcal{C}}\left(\begin{tikzcd} & B & B' \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rrr, "h"'] & & & C \end{tikzcd} \right) \xrightarrow{\Phi} \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\shortparallel"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd} \right)$$

$$\mathcal{C}/\mathrm{Ker}(\Phi)$$

This is undefined!

What should be identified? $B = B'$, $g \circ f \cancel{=} h$

Counter example :

$$
\boxed{\mathcal{C}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \boxed{\mathcal{D}}
$$

$$
\left(\begin{tikzcd}
 & B & B' \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rrr, "h"'] & & & C
\end{tikzcd}\right)
\xrightarrow{\ \Phi\ }
\left(\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[r, phantom, shift left=2ex, "\parallel" {xshift=3ex}] & & C
\end{tikzcd}\right)
$$

$$
\mathcal{C}/\mathrm{Ker}(\Phi) =
\left(\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[r, phantom, shift left=2ex, "\nparallel" {xshift=3ex}] & & C
\end{tikzcd}\right)
\xrightarrow{\ \Phi'\ } \mathcal{D}
$$

$$
\Rightarrow \left(\mathcal{C}/\mathrm{Ker}(\Phi)\right)/\mathrm{Ker}(\Phi') \cong \mathrm{Im}(\Phi)
$$

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Quotient by kernels

The kernel of f =
$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$\mathrm{Ker}(f) = \{(a, a') \mid f(a) = f(a')\}$

The quotient by Ker(f) = $\mathrm{Ker}(f) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \twoheadrightarrow \mathrm{CoEq}(\pi_1, \pi_2)$

The quotient $A/_{\mathrm{Ker}(f)}$

Morphisms obtained by this process are called regular epimorphisms.

A sequence of regular epimorphisms

$$
\begin{tikzcd}
 & \mathrm{Ker}(f) \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "f"] \arrow[d] & B \\
 & \mathrm{Ker}(f_1) \arrow[r, shift left] \arrow[r, shift right] & A_1 \arrow[ur, "f_1"'] \arrow[d] & \\
 A/\mathrm{Ker}(f) \arrow[urr, equal] & & A_2 \arrow[uur, "f_2"'] \arrow[d] & \\
 & & \vdots \arrow[d] & \\
 & \mathrm{Ker}(f_\omega) \arrow[r, shift left] \arrow[r, shift right] & \operatorname*{Colim}_{n<\omega} A_n \arrow[uuuur, "f_\omega"'] \arrow[d] & \\
 & & \vdots &
\end{tikzcd}
$$

The image of a morphism

= The smallest subobject
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "\exists g"'] & B \\ & B' \arrow[u, phantom, "\subseteq" sloped] \end{tikzcd}$$

$A \xrightarrow{g} B'$ factors through no proper $B'' \rightarrowtail B'$

$\iff$ $A \xrightarrow{g} B$ is a strong epimorphism.

if pullbacks exist

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "\forall m"] \\ B \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

A factorization
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\text{str.epi.}"'] & \circlearrowleft & B \\ & I \arrow[ur, tail, "\text{mono}"'] & \end{tikzcd}$$
is unique if exists.

The image $\mathrm{Im}(f)$

The fundamental theorem, categorically

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r] \arrow[d] & \mathrm{Im}(f) \arrow[r, tail] & B \\
& A/\mathrm{Ker}(f) \arrow[ur, "\cong"] \arrow[d] & & \\
& (A/\mathrm{Ker}(f))/\mathrm{Ker}(f_1) \arrow[uur] \arrow[d] & \vdots & \\
& \vdots & &
\end{tikzcd}$$

Remark

$$\{\text{Regular epis}\} \subseteq \{\text{strong epis}\}$$

If $\boxed{\{\text{Regular epis}\} = \{\text{strong epis}\}}$, we have $A/\mathrm{Ker}(f) \xrightarrow{\cong} \mathrm{Im}(f)$

$=$ **The fundamental theorem**

The approximated fundamental theorem

Proposition. [Kelly '69, Gabriel, Ulmer '71]

In a locally presentable category, the transfinite sequence stabilizes at some ordinal λ,

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[drrrrr, bend right, "f"'] & A_1 \arrow[r, two heads] \arrow[drrrr, bend right, "f_1"] & \cdots & A_\lambda \arrow[r, two heads, "\cong"] \arrow[drr, "f_\lambda"] & & A_{\lambda+1} \arrow[d] \\
& & & & & B
\end{tikzcd}$$

and gives an image factorization

$$\begin{tikzcd}
A \arrow[r, two heads, "\text{str.epi.}"'] & A_\lambda \arrow[r, tail, "f_\lambda", "\text{mono}"'] & B
\end{tikzcd}$$

$$A_\lambda \;\|\; \mathrm{Im}(f)$$

$$\underbrace{(A/\mathrm{Ker}(f))/\cdots}_{\lambda \text{ times}} \cong \mathrm{Im}(f)$$

The decomposition number

Definition. [Gabriel, Ulmer '71, Börger '91, KN.]

The regular (canonical) decomposition number $\delta(f)$ of $A \xrightarrow{f} B$ is the smallest α at which the seq. stabilizes ($A_{\alpha,\alpha+1}$: iso).

$$
\begin{tikzcd}
A \arrow[r, two heads, "A_{0,1}"] \arrow[drrrr, bend right, "f"'] & A_1 \arrow[r, two heads, "A_{1,2}"] \arrow[drrr, bend right, "f_1"] & A_2 \arrow[r, two heads, "A_{2,3}"] \arrow[drr, bend right, "f_2"] & A_3 \arrow[dr, bend right] & \cdots \\
 & & & & B
\end{tikzcd}
$$

The global regular (canonical) decomposition number $\delta(\mathcal{C})$ of a category $\mathcal{C}$ is the smallest γ s.t. $\delta(f) \leq \gamma$ for any f in $\mathcal{C}$.

Examples

- Grp, Ring, Module_R : $\delta = 2$
- Any regular category : $\delta \leq 2$

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "f"] \arrow[d, two heads] & B \\
& A/\mathrm{Ker}(f) \arrow[r, phantom, "\cong"] & \mathrm{Im}(f) \arrow[u, tail]
\end{tikzcd}$$

- $\delta(\mathbf{Pos}) = 2$
- $\delta(\mathbf{Field}) = 1$ $\quad (\because \quad K \cong \mathrm{Im}(f))$
- $\delta(\mathbf{Cat}) = 3$ ([Bednarczyk, et.al. '99])

Any strong epi is the composite of two regular epis.

The decomposition number (ctn.)

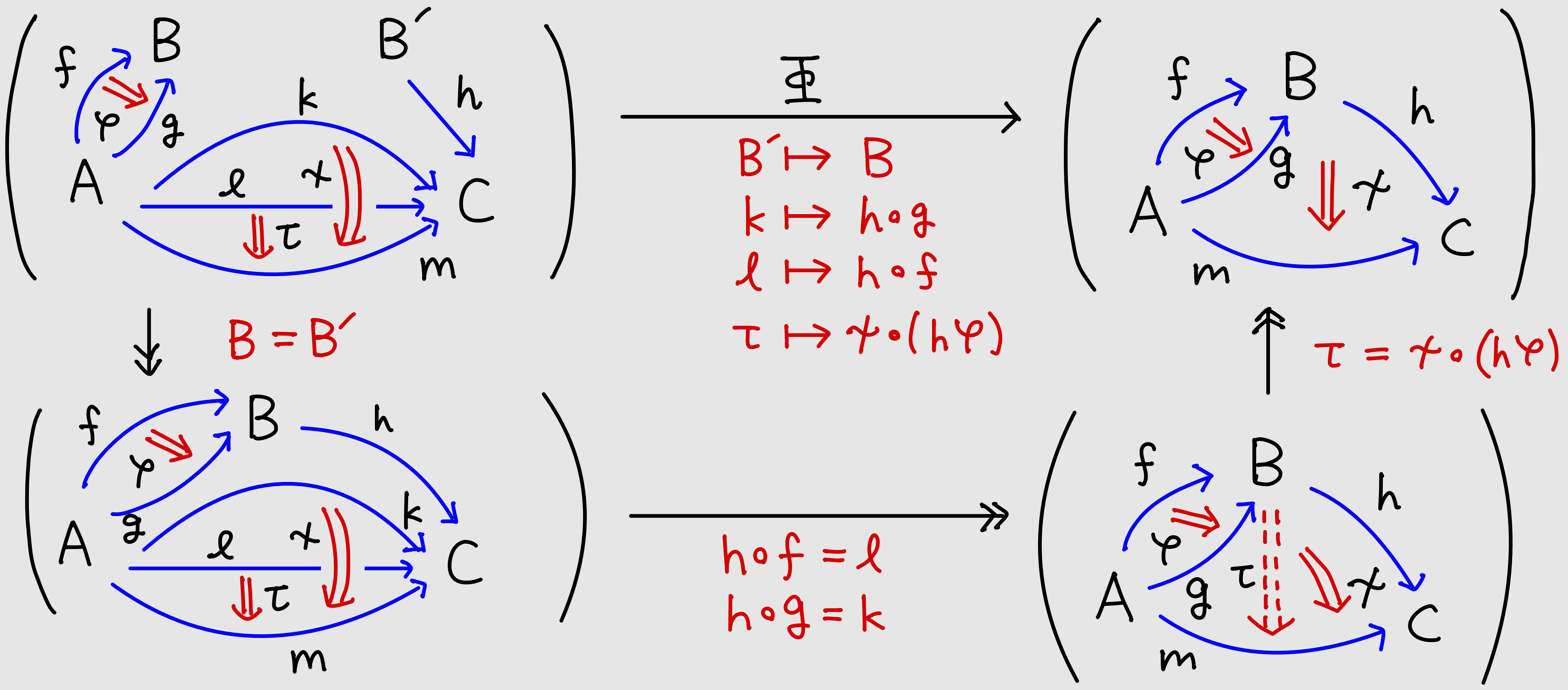
Proposition. [Gabriel, Ulmer '71, KN.]

$\mathcal{C}$: locally λ-presentable $\Rightarrow$ $\delta(\mathcal{C}) \leq \lambda + 1$.

In particular,

$\mathcal{C}$: locally finitely presentable $\Rightarrow$ $\delta(\mathcal{C}) \leq \omega + 1$.

Finding the exact δ



$\delta(2\text{-Cat}) = 4$?

$\delta(n\text{-Cat}) = ?$

$\delta(\text{DblCat}) = ?$

$\delta(\text{PolyCat}) = ?$

. . .

Ideal : all the solutions at once !

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Generalized algebraic theory (GAT)

[Cartmell '78]

GAT is an extension of algebraic theories in which you can use dependent sorts (types).

e.g. $\mathbb{T}_{\mathrm{Cat}}$

Sorts

$$\vdash \mathrm{Ob}$$

$$x : \mathrm{Ob},\ y : \mathrm{Ob} \vdash \mathrm{Mor}(x,y)$$

Operators

$$x : \mathrm{Ob} \vdash \mathrm{id}(x) : \mathrm{Mor}(x,x)$$

$$x, y, z : \mathrm{Ob},\ f : \mathrm{Mor}(x,y),\ g : \mathrm{Mor}(y,z) \vdash \underbrace{\mathrm{comp}(x,y,z,f,g)}_{g \circ f} : \mathrm{Mor}(x,z)$$

Axioms

$$x : \mathrm{Ob},\ f : \mathrm{Mor}(x,y) \vdash \underbrace{\mathrm{comp}(x,x,y,\mathrm{id}(x),f) = f}_{f \circ \mathrm{id}_x = f} : \mathrm{Mor}(x,y)$$

...

Using generalized algebraic theory

Proposition. [Cartmell '78, Adámek, Rosický '94]

$\mathcal{C}$: locally finitely presentable $\Leftrightarrow$ $\exists\, \mathbb{T}$: GAT $\quad \mathcal{C} \simeq \mathrm{Mod}(\mathbb{T})$

Given a GAT $\mathbb{T}$, how can we know $\delta(\mathrm{Mod}(\mathbb{T}))$?

Observation.

$$\left(\begin{tikzcd} & B & B' \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rrr, "h"'] & & & C \end{tikzcd}\right) \xrightarrow{\ \Phi\ } \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\parallel"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right)$$

$B = B'$ $\searrow$ $\qquad$ $\nearrow$ $g \circ f = h$

$$\left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\neq"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right)$$

Identifying non-dependent terms.

Identifying terms of "dependency rank 1".

The dependency rank $\rightsquigarrow \delta$

Definition. In a GAT $\mathbb{T}$, we define the dependency rank $dr(A)$ for each well-formed type $\Gamma \vdash A$ by

$$dr(A) = \max\{dr(B) \mid B \text{ appear in } \Gamma\} + 1.$$

Example. $dr(\mathrm{Ob}) = 1$, $dr(\mathrm{Mor}(x,y)) = 2$

Theorem. [KN.] $\mathbb{T}$: non-descending, $\max_{A \text{ in } \mathbb{T}} dr(A) \leq n$

$\Rightarrow \forall g : M \to N$ in $\mathrm{Mod}(\mathbb{T})$ $\delta(f) \leq n+1$, hence $\delta(\mathrm{Mod}(\mathbb{T})) \leq n+2$

We call $\mathbb{T}$ a non-descending if

- $\Gamma \vdash f(\vec{x}) : B$: operator symbol
- $\Gamma \vdash t = s : B$: axiom

$\Rightarrow dr(B) \geq \max_{A \text{ in } \Gamma} dr(A)$

Examples

- $\mathrm{nCat} = \mathrm{Mod}(\mathbb{T}_{\mathrm{ncat}})$

Sorts $\vdash \underline{0}$, $x, x' : \underline{0} \vdash \underline{1}(x,x')$, $x, x' : \underline{0}, y, y' : \underline{1}(x,x') \vdash \underline{2}(x,x',y,y')$ ...

Operators $x, x' : \underline{0}$, $y, y', y'' : \underline{1}(x,x')$, $z : \underline{2}(x,x',y,y')$, $z' : \underline{2}(x,x',y',y'')$

$\vdash \text{v-comp}(x,x',y,y',y'',z,z') : \underline{2}(x,x',y,y'')$ ...

$\mathrm{dr}(\underline{n}) = n+1$, so $\delta(\mathrm{nCat}) \leq n+2$. In fact, $\delta(\mathrm{nCat}) = n+2$.

- $\delta(\mathrm{strMonCat}) = 3$.

- $\delta(\mathrm{strDblCat}) = 4$.

not non-descending!

- $\exists\ \mathbb{T}$: GAT with $\max\{\mathrm{dr}(A)\} = 1$. $\delta(\mathrm{Mod}(\mathbb{T})) = \omega + 1$.

Summary

- The fundamental theorem of homomorphisms states that

  strong epis $\Leftrightarrow$ regular epis

- They do not coincide in general, but in l.p. categories.

  strong epis $\Leftrightarrow$ transfinite composites of regular epis.

- The number of regular epis needed (= decomposition number) can have an upper bound determined by the corresponding GAT, which is the dependency rank $+ 2$ (when non-descending).

Omitted topics ([KN.])

- The regular decomposition gives the most efficient way to present a str.epi as a composite of reg.epis.

$$\xrightarrow{f} \;=\; \underbrace{\overset{f_1}{\twoheadrightarrow}\ \overset{f_2}{\twoheadrightarrow}\ \cdots\ \overset{f_n}{\twoheadrightarrow}}_{\text{reg. epis}} \implies \delta(f) \leq n$$

- Generalization of the class of regular epimorphisms

(localization functors in **Cat**, etc.)

- Finding an upper bound of δ using a presentation as models of a partial Horn theory [PV07].

References

[AHT89] Jií Adámek, Horst Herrlich, and Walter Tholen. Monadic decompositions. *J. Pure Appl. Algebra*, 59(2):111–123, 1989.

[ARV11] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic theories*, volume 184 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011. A categorical introduction to general algebra, With a foreword by F. W. Lawvere.

[B"91] Reinhard Börger. Making factorizations compositive. *Comment. Math. Univ. Carolin.*, 32(4):749–759, 1991.

[BBP99] Marek A. Bednarczyk, Andrzej M. Borzyszkowski, and Wieslaw Pawlowski. Generalized congruences—epimorphisms in *Cat*. *Theory Appl. Categ.*, 5:No. 11, 266–280, 1999.

[Car86] John Cartmell. Generalised algebraic theories and contextual categories. *Ann. Pure Appl. Logic*, 32(3):209–243, 1986.

[Fre25] Jonas Frey. Duality for clans: an extension of gabriel–ulmer duality. *J. Symb. Log.*, pages 1–38, 2025.

[GU71] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*, volume Vol. 221 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1971.

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium. Vol. 2*, volume 44 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, Oxford, 2002.

[Joy17] André Joyal. Notes on clans and tribes. *arXiv preprint arXiv:1710.10238*, 2017.

[Kaw24] Yuto Kawase. Relativized universal algebra via partial horn logic, 2024.

[MS82] John L. MacDonald and Arthur Stone. The tower and regular decomposition. *Cahiers Topologie Géom. Différentielle*, 23(2):197–213, 1982.

[MT82] John MacDonald and Walter Tholen. Decomposition of morphisms into infinitely many factors. In *Category theory (Gummersbach, 1981)*, volume 962 of *Lecture Notes in Math.*, pages 175–189. Springer, Berlin-New York, 1982.

[PV07] E. Palmgren and S. J. Vickers. Partial Horn logic and cartesian categories. *Ann. Pure Appl. Logic*, 145(3):314–353, 2007.

Thank you !

arxiv:2604.05744

hayatonasu.github.io

I will talk more about GAT (clans) at FMCS ☺

A1/A2

Why do we take the strict upper bound?

$\delta(\mathscr{C}) = \omega$ $\Longleftrightarrow$ For any $n < \omega$, there is f in $\mathscr{C}$ s.t. $\delta(f) \geq n$, but every f in $\mathscr{C}$ has $\delta(f) < \omega$.

arbitrarily large finite

$\delta(\mathscr{C}) = \omega + 1$ $\Longleftrightarrow$ There is f in $\mathscr{C}$ s.t. $\delta(f) = \omega$, and that is the upper bound.

countable

$$
\begin{tikzcd}
A \arrow[r, two heads] & A_1 \arrow[r, two heads] & A_2 \arrow[r, two heads] & \cdots \arrow[r, two heads] & A_n \arrow[r, two heads] \arrow[drr, "\ncong"'] & \cdots \arrow[r, two heads] & \operatorname*{colim}_{n<\omega} A_n \arrow[d, "\cong"] \\
 & & & & & & \mathrm{Im}(f)
\end{tikzcd}
$$

$\times$ $\delta(\mathscr{C}) = \omega$

$\checkmark$ $\delta(\mathscr{C}) = \omega + 1$

Misc

$\Phi : A \to B$ is a str. epi $\Leftrightarrow$ surj. on obj. + $\forall g$ in B $\exists f_1, \ldots, f_n$

$$g = \Phi(f_n) \circ \cdots \circ \Phi(f_1)$$