Double categories of relations relative to factorization systems

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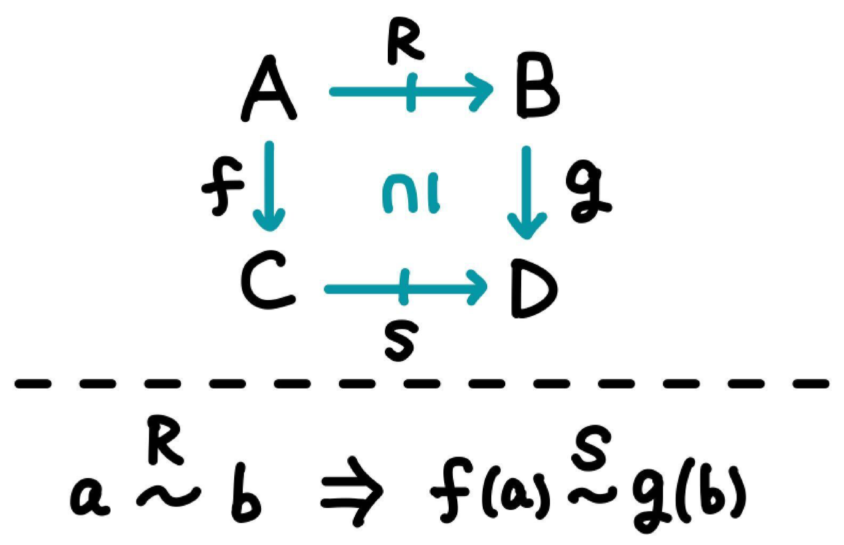
WHAT IS A DOUBLE CATEGORY?

Double categories generalize categories in that they have two different kinds of morphisms called **vertical** and **horizontal arrows,** and cells framed by those arrows.

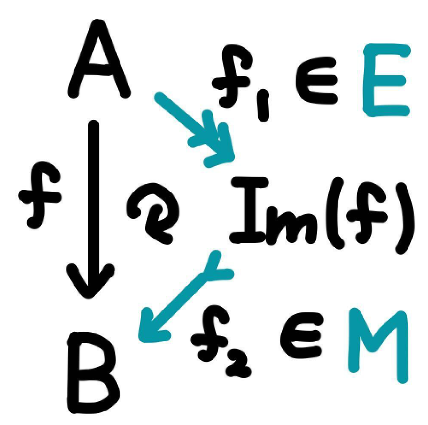
A typical example is the double category $\mathbf{Rel}$ of **sets**, **functions**, and **(binary) relations**.



WHAT IS A FACTORIZATION SYSTEM?

In the category of sets, a function can be **factored into** a surjection and an injection **essentially uniquely**, and similar situations happen for other structures.

An abstract framework for this is known as **factorization systems** (E,M) where E and M are classes of morphisms corresponding to those of surjections and injections.



THE AIM OF THIS STUDY

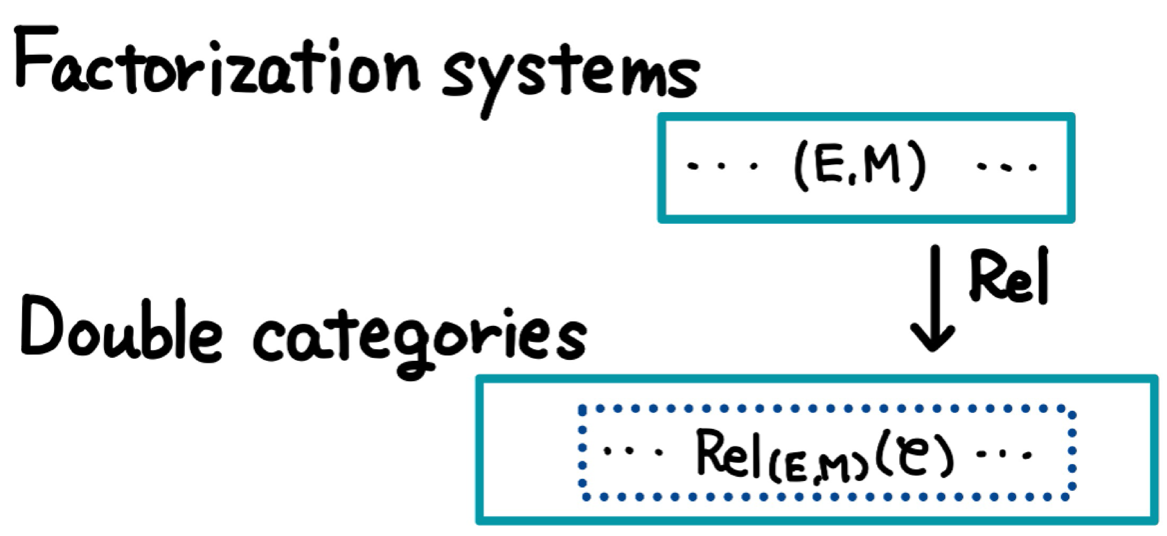
For any category C with finite products and a factorization system (E,M), an **M-relation** A ⇸ B is defined as an arrow into A×B belonging to M.

$A \overset{R}{\nrightarrow} B \quad | \quad R \rightarrowtail A\times B \in M$

When the factorization system is well behaved with pullbacks, we can construct the double category $\mathbf{Rel}_{(E,M)}(C)$ whose horizontal arrows are M-relations.

The problem is:

what axiomatic conditions characterize the double categories arising in this way?



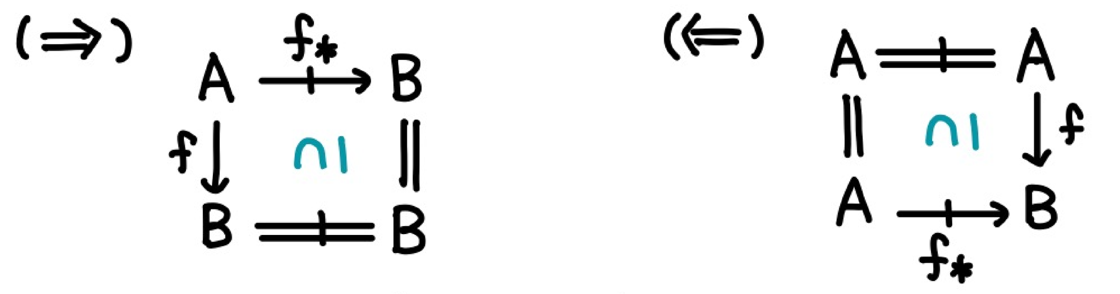
WHAT ARE THE CHARACTERISTICS OF DOUBLE CATEGORIES OF RELATIONS?

1. From functions to relations: graphs of functions as relations.

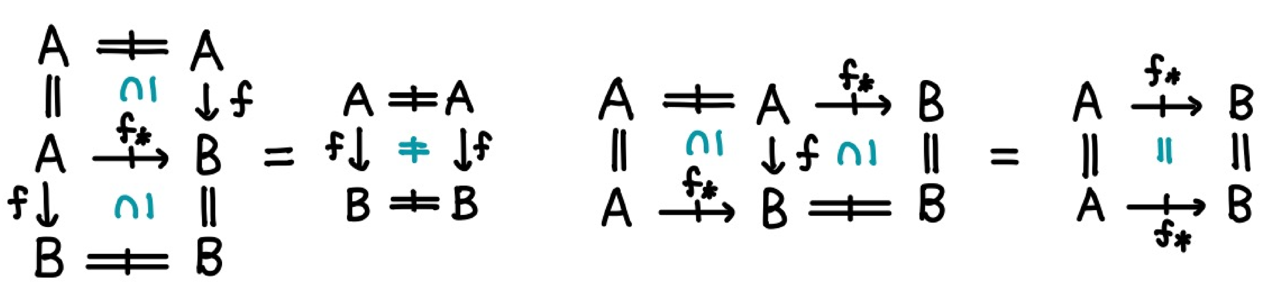
A function $f : A \to B$ defines $f_* : A \nrightarrow B$ as

$$a \overset{f_*}{\sim} b \overset{def}{\iff} f(a) = b.$$

The two directions of the ⇔ can be understood as



We can compose these cells in two ways, which are trivially equal to the identity cells.



Definition. A double category is called ***an equipment*** if every vertical arrow f: A→B admits horizontal arrows f_*: A⇸B and f^*: B⇸A together with cells (as shown above) satisfying certain coherence conditions.

2. Products of sets and intersections of relations.

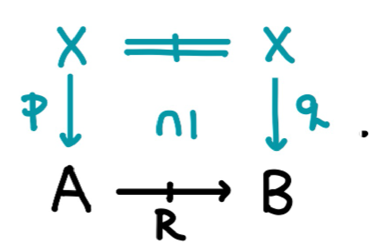
Definition. An equipment is said to be ***cartesian*** if it has *"double-categorical finite products"*.

Cartesian equipments are known to admit finite products of objects and those of horizontal arrows.

3. From relations to functions: comprehension of relations as pairs of functions.

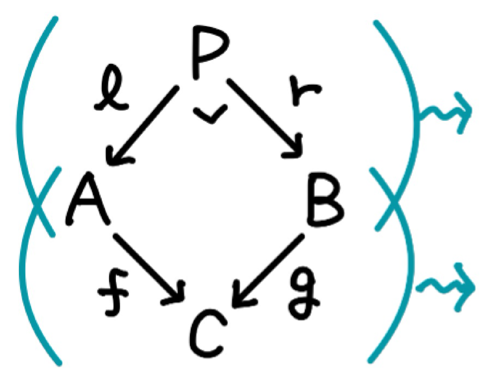
For a relation $R : A \nrightarrow B$ we can define

This gives the universal pair of functions among those admitting



Definition. **A tabulator** of R:A⇸B is the universal pair of functions p: X→A and q: X→B with a cell as shown. In an equipment, a tabulator is called **strong** if $p^*q_*\cong R$ canonically.

4. The relation defined by the equation f(a) = g(b) is the same as the one determined by the pullback of f and g.



$a \sim_1 b \overset{def}{\iff} \exists p \in P \;\; l(p) = a,\ r(p) = b$

$a \sim_2 b \overset{def}{\iff} f(a) = g(b)$

$\implies \sim_1 = \sim_2 : A \nrightarrow B$

Definition. An equipment is said to have **Beck-Chevalley pullbacks** if its vertical part has arbitrary pullbacks, and for any pullback square consisting of (l,r,f,g) as shown on the left, we have $l^*r_*\cong f_*g^*$ canonically.

MAIN RESULT

Theorem. A double category $\mathbf{D}$ is equivalent to $\mathbf{Rel}_{(E,M)}(C)$ for some category C with finite limits and a stable factorization system (E,M) on C iff

- $\mathbf{D}$ is a cartesian equipment,
- $\mathbf{D}$ has strong tabulators, and
- $\mathbf{D}$ has Beck-Chevalley pullbacks, and
- Fib($\mathbf{D}$) (whose definition not mentioned) is closed under composition.

The class Fib($\mathbf{D}$) is the right class of the corresponding factorization system.

ADDITIONAL REMARKS

1. We do not assume M to be a class of monic arrows, or E to be a class of epic arrows. Each of these conditions for (E,M) can also be characterized by double-categorical conditions.
2. The class of vertical arrows Fib($\mathbf{D}$) is defined via tabulators.
3. In the cases of M being the class of all arrows and all monic maps, we obtain the double categories of spans $\mathbf{Span}(C)$ and that of (ordinary) relations $\mathbf{Rel}(C)$, whose characterizations were already known and can be recovered from our results.

For more details, see the supplementary material.

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This poster and supplementary materials are available via THIS QR CODE. →

AARMS Atlantic Association for Research in the Mathematical Sciences

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DOUBLE CATEGORIES OF RELATIONS RELATIVE TO FACTORIZATION SYSTEMS (A SUPPLEMENTARY NOTE)

KEISUKE HOSHINO AND HAYATO NASU

1. Definitions

Definition 1. By a *(pseudo-)double category* $\mathbb{D}$, we mean a pseudo-category in the 2-category $\mathcal{CAT}$ of locally small categories. In other words, a double category consists of two (locally small) categories $\mathbb{D}_0$, $\mathbb{D}_1$ and functors

$$\mathbb{D}_1\ {}_{\mathsf{tgt}}\!\times_{\mathsf{src}} \mathbb{D}_1 \xrightarrow{\ \odot\ } \mathbb{D}_1 \underset{\mathsf{tgt}}{\overset{\mathsf{src}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\ \mathtt{Id}\ }}}}} \mathbb{D}_0\ .$$

These data come equipped with natural isomorphisms that stand for the associativity law and the unit laws.

Objects and arrows of $\mathbb{D}_0$ are called ***objects*** and ***vertical arrows*** of the double category $\mathbb{D}$. We use the notation $f \fatsemi g$ for the composition of $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{D}_0$. An object p of $\mathbb{D}_1$ whose values of src and tgt are A and B, respectively, is called a ***horizontal arrow***[1] from A to B, and written as $p\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$. We use the notation $p \odot q$, or simply pq, for the composite of $p\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ and $q\colon B \relbar\joinrel\mapstochar\joinrel\rightarrow C$ in $\mathbb{D}_1$. An arrow $\alpha\colon p \longrightarrow q$ in $\mathbb{D}_1$ is called a ***double cell*** (or merely a ***cell***) in the double category $\mathbb{D}$. This cell is drawn as below, where $\mathsf{src}(\alpha) = f$ and $\mathsf{tgt}(\alpha) = g$.

$$\begin{array}{ccc} A & \xrightarrow{\ p\ } & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\ q\ ]{} & D \end{array} \tag{1}$$

■

Definition 2 ([HN25, Definition 2.2.1])**.** An orthogonal factorisation system (OFS) on a category $\mathbf{C}$ consists of a pair (E, M) of classes of arrows in $\mathbf{C}$ that satisfies the following conditions:

i) E and M are closed under composition and contain isomorphisms.
ii) Every arrow $e\colon X \longrightarrow Y$ in E is left orthogonal to every arrow $m\colon A \longrightarrow B$ in M; that is, every commutative square

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle e}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle m} \\ Y & \longrightarrow & B \end{array}$$

has a unique diagonal filler $f\colon Y \longrightarrow A$ that makes two triangles commute.
iii) Every arrow f in $\mathbf{C}$ factors as $f = e \fatsemi m$ where e belongs to E and m belongs to M.

E and M are called ***the left class*** and ***the right class*** of the OFS, respectively. For a factorisation of f as $f = e \fatsemi m$, we say m is the M-***image*** of f if $e \in \mathrm{E}$ and $m \in \mathrm{M}$.

An OFS (E, M) is a ***stable orthogonal factorisation system (SOFS)*** if E is stable under pullback. We only treat orthogonal factorisation systems, and so we omit the adjective 'orthogonal' in the sequel. ■

Date: May 8, 2025.

[1] Note that in a significant portion of the literature, vertical arrows are referred to as what we designate as horizontal arrows, and vice versa. The difference cannot be dismissed since only one class of arrows requires strict associativity and unit laws.

2. Main result and additional remarks

Definition 3 ([HN25, Definition 3.3.2])**.** In a double category $\mathbb{D}$ with the vertical terminal 1, we say a vertical morphism $f\colon A \longrightarrow X$ is a ***fibration*** if there exists a horizontal arrow $p\colon X \twoheadrightarrow 1$ and a tabulating cell

$$\begin{array}{ccc} & A & \\ {}^{f}\swarrow & \texttt{tab} & \searrow^{!} \\ X & \xrightarrow[p]{\quad\shortmid\quad} & 1 \end{array} .$$

We write $\mathrm{Fib}(\mathbb{D})$ for the class of fibrations in $\mathbb{D}$. ■

Theorem 4 ([HN25, Full version, Theorem 3.3.20])**.** *The following are equivalent for a double category* $\mathbb{D}$*.*

- *i)* $\mathbb{D}$ *is equivalent to* $\mathbb{R}\mathrm{el}_{\mathrm{E},\mathrm{M}}(\mathbf{C})$ *for some category* $\mathbf{C}$ *with finite limits and a stable factorisation system* (E, M) *on* $\mathbf{C}$*.*
- *ii)* $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and an* M*-comprehension scheme for some stable system* M *on* $\mathbf{V}(\mathbb{D})$*.*
- *iii)* $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and a left-sided* M*-comprehension scheme for some stable system* M *on* $\mathbf{V}(\mathbb{D})$*.*
- *iv)* $\mathrm{Fib}(\mathbb{D})$ *is closed under composition, and* $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and strong tabulators.*
- *v)* $\mathrm{Fib}(\mathbb{D})$ *is closed under composition, and* $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and left-sided strong tabulators.*

In relation to the additional remarks section in the poster, we have the correspondence Table 1.

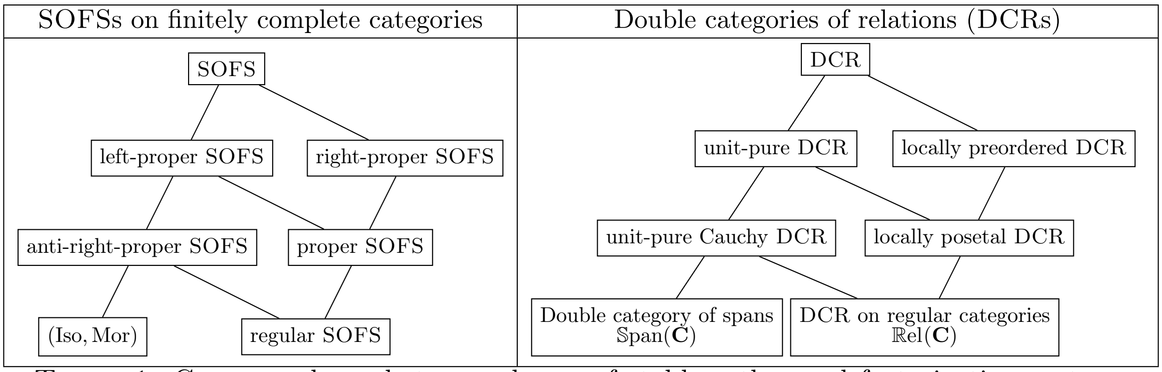

Table 1. Correspondence between classes of stable orthogonal factorization systems and double categories of relations.

Definition 5 ([HN25, Definition 2.2.1])**.** We say it is ***right-proper*** if M is a subclass of the class of all monomorphisms, ***left-proper*** if E is a subclass of the class of epimorphisms, and ***proper*** if it is both right-proper and left-proper. Furthermore, we call it ***anti-right-proper*** if the class of monomorphisms is a subclass of M. ■

Definition 6 ([Ale18, Definition 4.3.7],[HN25, Definition 4.1.1])**.** A double category $\mathbb{D}$ is called ***unit-pure*** if $\mathtt{Id}\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ is fully faithful. In more concrete terms, $\mathbb{D}$ is unit-pure if every cell of the form

$$\begin{array}{ccc} X & \xrightarrow{\mathtt{Id}_X} & X \\ {\scriptstyle f}\downarrow & \cdot & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\mathtt{Id}_Y]{} & Y \end{array}$$

is an identity cell, implying f and g are equal. ■

Definition 7 ([HN25, Definition 4.1.7])**.** Let $\mathbb{D}$ be a double category. We say that $\mathbb{D}$ is ***locally preordered*** if there must be at most one cell framed by a pair of vertical arrows and a pair of horizontal arrows. We say that $\mathbb{D}$ is ***locally posetal*** if it is locally preordered and the vertical 2-category $\mathcal{V}(\mathbb{D})$ is locally posetal. ■

In some papers [GP99, Ště24], a locally preordered double category is called a ***flat double category***.

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