These slides have errors: The pullbacks in the page 18 and the appendix may not be correct, what we have is the pullback along the forgetful functor from the 2-category of cartesian equipments to the 2-category of cartesian fibrational virtual double categories. The correct statement can be found in [1].

References

[1] Hayato Nasu. Logical Aspects of Virtual Double Categories. Master's thesis, Kyoto University, 2025. `https://hayatonasu.github.io/hayatonasu/Papers/masterthesisupdated.pdf`

(Hyper)doctrines as Virtual Double Categories

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Introduction

A (hyper)doctrine = a pseudo-functor $P : \mathcal{C}^{op} \longrightarrow \mathbf{Preorder}$.

$$\begin{tikzcd}
n:\mathrm{Nat},\ n':\mathrm{Nat} \arrow[r, mapsto] & \boxed{n=n'+1} \arrow[r] \arrow[d] & \boxed{n>n'} & \cdots \\
m:\mathrm{Nat} \arrow[u, "{[m/n,\ 2m/n']}"] \arrow[r, mapsto] & \boxed{2m=m+1} \arrow[r] & \boxed{2m>m} & \cdots
\end{tikzcd}$$

A fibration $\doteq$ a pseudo-functor $P : \mathcal{C}^{op} \longrightarrow \mathbf{CAT}$

= a proof relevant doctrine

$$\begin{tikzcd}
 & \boxed{\text{Regular categories}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{Fibrations}} \arrow[rr, leftrightarrow, bend right, "\text{Direct connection?}"'] & & \boxed{\text{Double categories}} \\
 & \boxed{\text{Factorization systems}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Structures

1. Double categories of relations relative to factorisation systems
   1.1. Double categories : definition and examples
   1.2. A characterization theorem

2. Virtual double categories and fibrations.
   2.1. Making fibrant VDCs from primary fibrations.
   2.2. $\exists$ in virtual double categories.

The 1st half is based on joint work with Keisuke Hoshino available on ArXiv:
Hoshino, N., "Double categories of relations relative to factorisation systems" (2023)

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Double categories

A **double category** $\mathbb{D}$ consists of the following data :

- **objects** $A, B, C, \ldots$
- **vertical arrows** $\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$, … and their composition

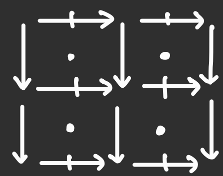
$$\begin{tikzcd} \bullet \arrow[dd, "g \circ f"'] \\ \\ \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[d, "f"'] \\ \bullet \arrow[d, "g"'] \\ \bullet \end{tikzcd}$$

- **horizontal arrows** $\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \end{tikzcd}$, … and their composition

$$\begin{tikzcd} \bullet \arrow[r, "R \circ S", "\shortmid"{marking}] & \bullet \end{tikzcd} := \begin{tikzcd} \bullet \arrow[r, "R", "\shortmid"{marking}] & \bullet \arrow[r, "S", "\shortmid"{marking}] & \bullet \end{tikzcd}$$

- **cells** $\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$, …

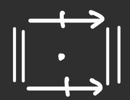
with appropriate data of composition and axioms.



In particular,

- (objects, vertical arrows) gives a category (the vertical category), and
- (objects, horizontal arrows, horizontal cells) gives a bicategory (the horizontal bicategory)

$\begin{tikzcd} \circ \arrow[r, "\shortmid"{marking}] & \circ \end{tikzcd}$



Examples of double categories

Rel : the double category of sets, functions, and relations

composition of horizontal arrows :

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, "S", "\shortmid"{marking}] & C \end{tikzcd}$$

$$:= \left\{ (a,c) \;\middle|\; \exists b : B .\ \begin{matrix} (a,b) \in R \\ (b,c) \in S \end{matrix} \right\}$$

identity horizontal arrows :

$$\begin{tikzcd} A \arrow[r, "\shortmid"{marking}] & A \end{tikzcd}$$

$$:= \{ (a, a') \mid a = a' \}$$

cells :

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\wedge\!|"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$$

$$\iff \quad (a,c) \in R \Rightarrow (fa, gc) \in S$$

Span : the double category of sets, functions, and spans

> A span from A to B consists of
> $$\begin{tikzcd} A & R \arrow[l, "l_R"'] \arrow[r, "r_R"] & B \end{tikzcd}$$

composition of horizontal arrows :

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, "S", "\shortmid"{marking}] & C \end{tikzcd}$$

$$:= \begin{tikzcd} & & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\ A & R \arrow[l, "l_R"'] \arrow[dr, "r_R"'] & & S \arrow[dl, "l_S"] \arrow[r, "r_S"] & C \\ & & B & & \end{tikzcd}$$

cells :

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & R \arrow[l, "l_R"'] \arrow[r, "r_R"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ B & S \arrow[l, "l_S"] \arrow[r, "r_S"'] & D \end{tikzcd}$$

Features of double categories of relations / spans

- A restriction $R(f;g)$ of
$$\begin{tikzcd} A \arrow[r, "R(f;g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart.}"] & C \arrow[d, "g"] \\ B \arrow[r, "R"', "\shortmid"{marking}] & D \end{tikzcd}$$
is the universal cell of this form:

$$\begin{tikzcd} X \arrow[r, "Q", "\shortmid"{marking}] \arrow[d, "h"'] & Y \arrow[d, "k"] \\ A \arrow[r, phantom, "\alpha"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "R"', "\shortmid"{marking}] & D \end{tikzcd} = \begin{tikzcd} X \arrow[r, "Q", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\exists!\,\tilde{\alpha}"] & Y \arrow[d, "k"] \\ A \arrow[r, "R(f;g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart.}"] & C \arrow[d, "g"] \\ B \arrow[r, "R"', "\shortmid"{marking}] & D \end{tikzcd}.$$

In $\mathbb{R}\mathrm{el}$,
$$\begin{tikzcd} A \arrow[r, "R(f;g)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\wedge\!\mathrm{I}"] & C \arrow[d, "g"] \\ B \arrow[r, "R"', "\shortmid"{marking}] & D \end{tikzcd} := \{(a,c) \mid (fa, gc) \in R\}$$

- A double category $\mathbb{D}$ is cartesian ($=$ has finite products) if $\mathbb{D} \xrightarrow{\Delta} \mathbb{D}^n$ has a right adjoint in $\mathbb{D}\mathrm{bl}$. $\mathbb{R}\mathrm{el}$ is cartesian with the obvious cartesian structure.

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Double categories of relations relativized

Relations are a special kind of spans : jointly monic spans.

$$A \xleftarrow{\ell} R \xrightarrow{r} B \text{ is a relation} \iff R \xrightarrow{\langle \ell, r\rangle} A \times B \in \underline{\mathrm{Mono}}$$

Replacing $\underline{\mathrm{Mono}}$ with other classes of morphisms, we have a variety of notions of relations.

An **orthogonal factorization system (OFS)** on a category $\mathcal{C}$ is a pair of classes of morphisms (E, M) such that:

(i) E and M are closed under composition and contain iso's.

(ii) E and M are orthogonal :

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "E \ni"'] & \bullet \arrow[d, "\in M"] \\ \bullet \arrow[r] \arrow[ur, "\exists !"] & \bullet \end{tikzcd}$$

(iii) Every morphism in $\mathcal{C}$ is factored as $\bullet \underset{E}{\xrightarrow{\rotatebox{90}{$\in$}}} \bullet \underset{M}{\xrightarrow{\rotatebox{90}{$\in$}}} \bullet$.

It is called **stable** if E is stable under pullback.

Double categories of relations relativized

From an SOFS (E,M) on a category $\mathcal{C}$, we construct the double category $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ consisting of the objects, the arrows, and the M-relations.

$$
\begin{tikzcd}
I \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & K \arrow[d, "g"] \\
J \arrow[r, "S"', "|" marking] & L
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
I \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & R \arrow[l, "l_R"'] \arrow[r, "r_R"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & K \arrow[d, "g"] \\
J & S \arrow[l, "l_S"] \arrow[r, "r_S"'] & L
\end{tikzcd}
\quad \text{with} \quad
\left\{
\begin{array}{l}
\langle l_R, r_R \rangle : R \longrightarrow I \times K \\
\langle l_S, r_S \rangle : S \longrightarrow J \times L
\end{array}
\right. \in M
$$

The composite of $I \overset{R}{\nrightarrow} J \overset{Q}{\nrightarrow} K$ is defined in the following way.

(1) Take the pullback

$$
\begin{tikzcd}
& P \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
R \arrow[dr, "r_R"'] & & Q \arrow[dl, "l_Q"] \\
& J &
\end{tikzcd}
.
$$

$$
\begin{tikzcd}
& & P \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
& R \arrow[dl, "l_R"'] \arrow[dr, "r_R"] & & Q \arrow[dl, "l_Q"'] \arrow[dr, "r_Q"] & \\
I & & J & & K
\end{tikzcd}
$$

(2) Factorize as

$$
\begin{tikzcd}
P \arrow[r, "{\langle p, q \rangle}"] \arrow[dr, "\in E"'] & R \times Q \arrow[r, "l_R \times r_Q"] & I \times K \\
& P' \arrow[ur, "{\langle l', r' \rangle}"', "\in M"'] &
\end{tikzcd}
,
$$

and define $I \overset{R}{\nrightarrow} J \overset{Q}{\nrightarrow} K := I \overset{P'}{\nrightarrow} K$.

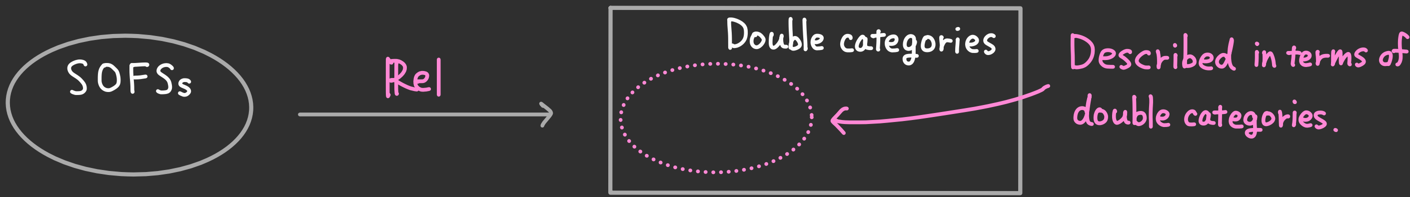
Double categories of relations relativized

Theorem [HN23] For a double category $\mathbb{D}$, the following are equivalent:

(1) $\mathbb{D} \simeq \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ for some SOFS (E,M) on some finitely complete category $\mathcal{C}$.

(2) $\mathbb{D}$ is a cartesian equipment with Beck-Chevalley pullbacks and admits the M-comprehension scheme.

If these hold, the classes M in these are the same.



$\mathcal{C}$	Regular categories	Lex categories	Quasitoposes
(E, M)	(RegEpi, Mono)	(Iso, All)	(Epi, StrongMono)
$\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$	DC of relations	DC of spans	DC of strong relations

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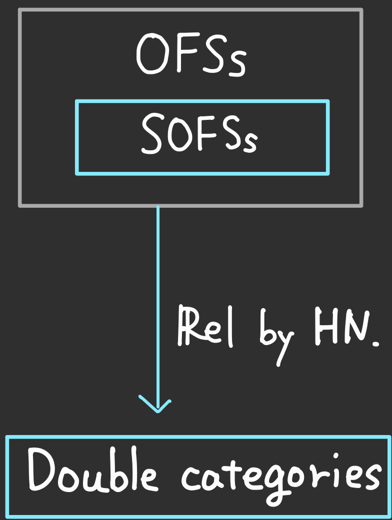
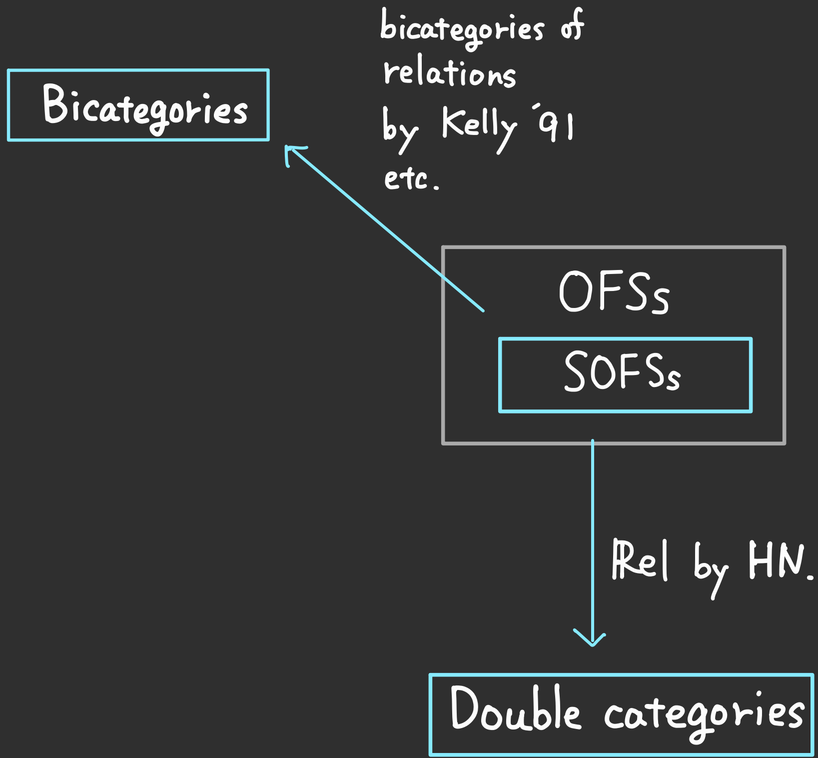
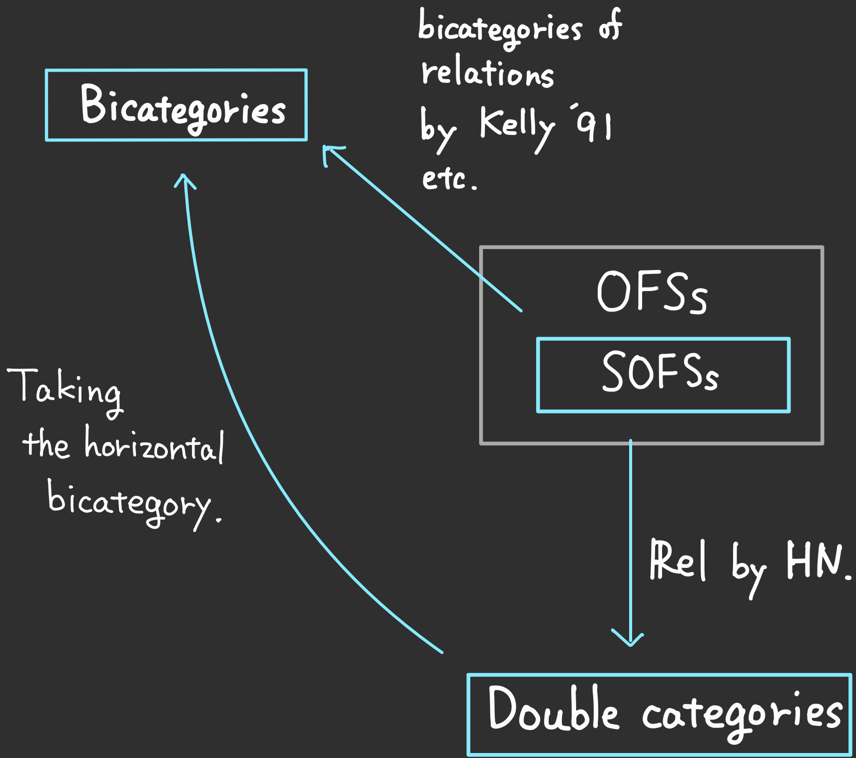
Some frameworks and their connection



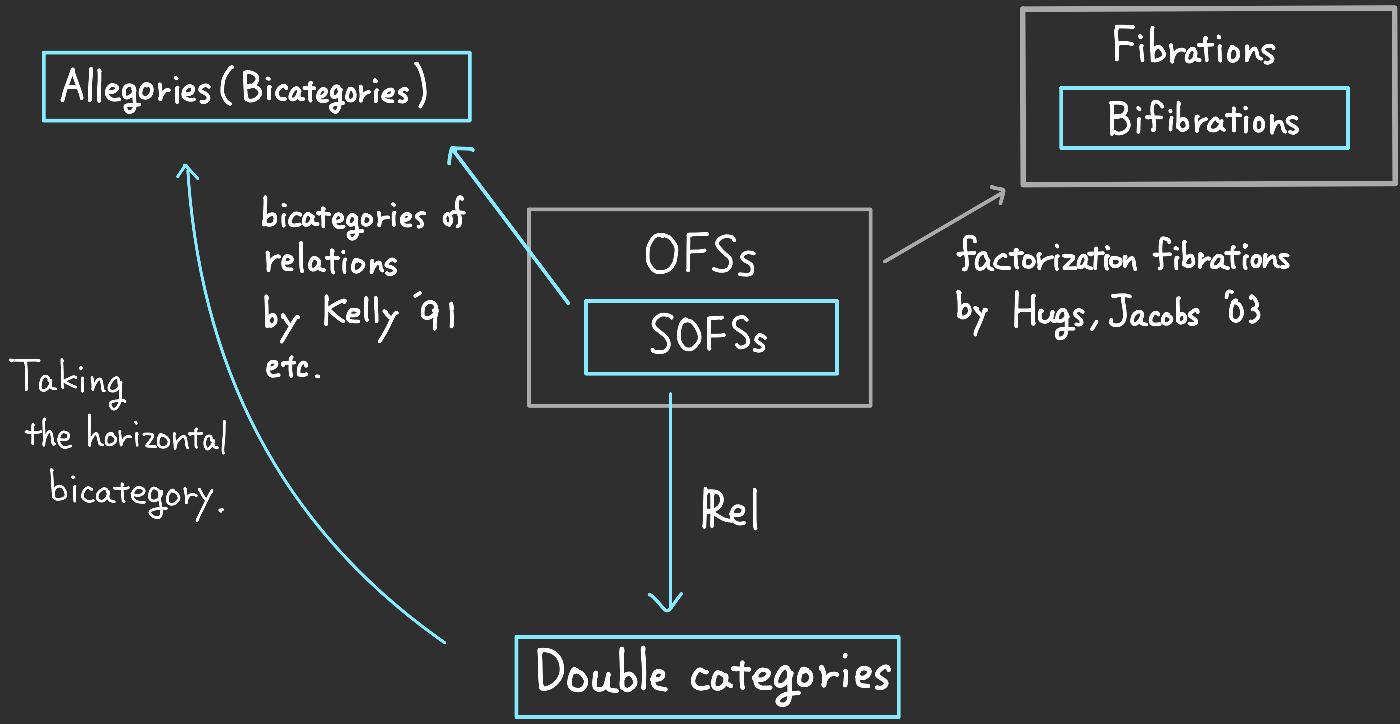
Some frameworks and their connection



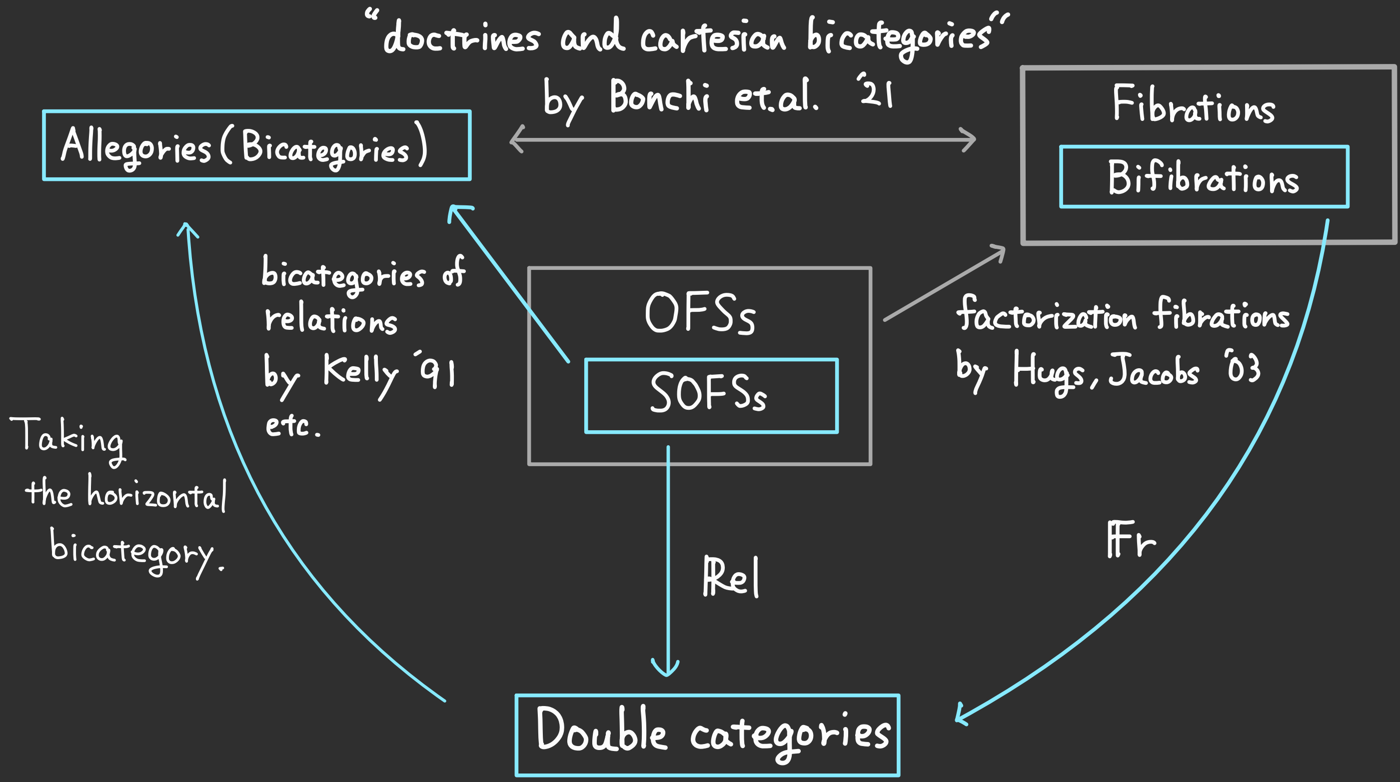
Some frameworks and their connection



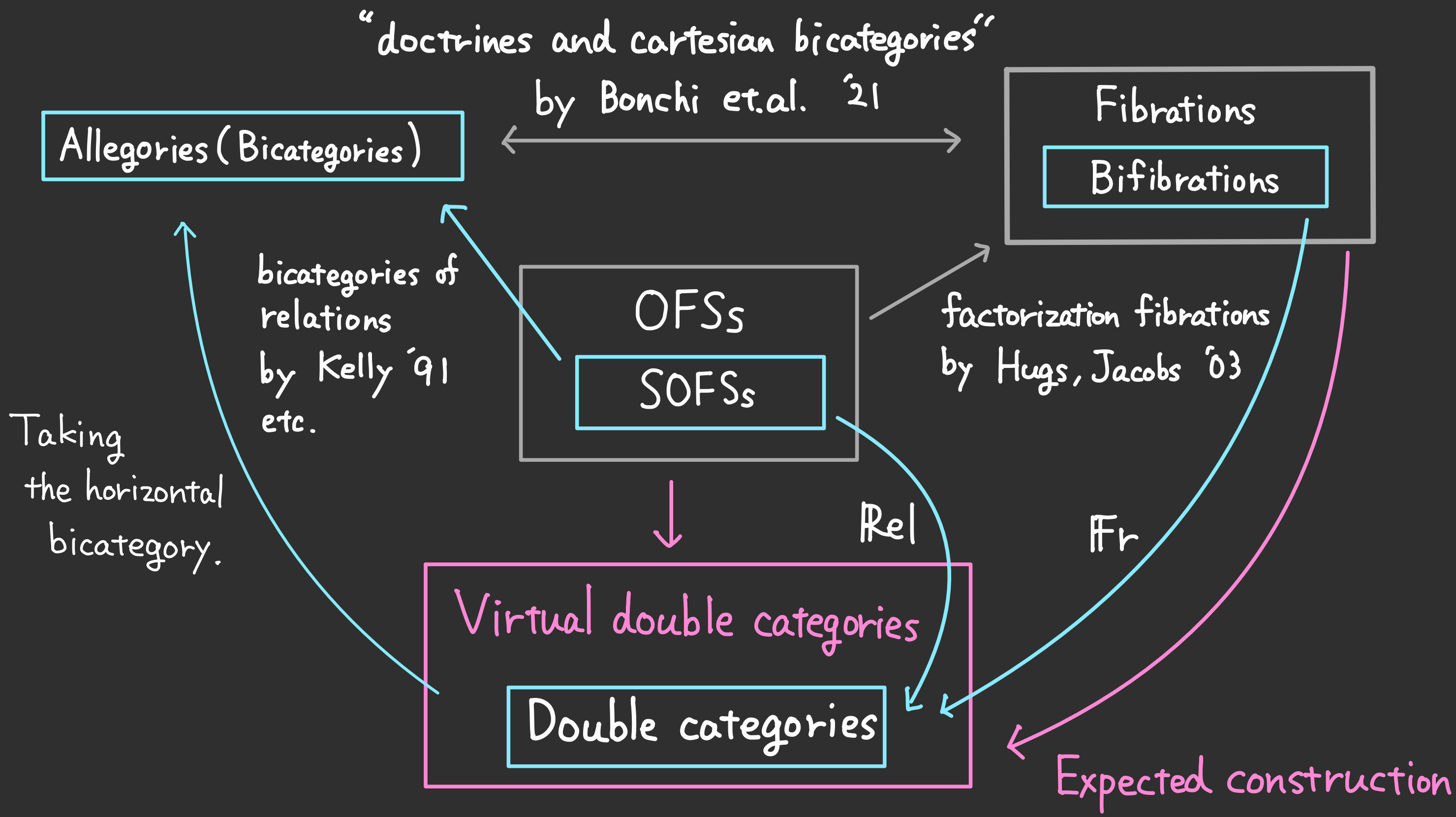
Some frameworks and their connection



Some frameworks and their connection



Some frameworks and their connection



Classes of fibrations

A fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is called primary if $\mathcal{B}$ and all the fibers $\mathcal{E}_I$ have finite products and all the reindexings $\mathcal{E}_J \xrightarrow{u^*} \mathcal{E}_I$ preserve them.

A primary fibration is called existential if the reindexings $\mathcal{E}_J \xrightarrow{\pi_J^*} \mathcal{E}_{I\times J}$ for all I and J have a left adjoint and they satisfy the Beck-Chevalley condition and the Frobenius reciprocity.

$$\begin{tikzcd}
\exists i.\, \varphi(i,j) & \varphi(i,j) \arrow[l, mapsto] \\
\mathcal{E}_J \arrow[r, bend right, ""{name=D}] & \mathcal{E}_{I\times J} \arrow[l, bend right, ""{name=U}] \arrow[from=U, to=D, phantom, "\perp"] \\
\psi(j) \arrow[r, mapsto] & \psi(j)
\end{tikzcd}$$

$$\frac{i:I,\ j:J \vdash \varphi(i,j) \Rightarrow \psi(j)}{j:J \vdash \exists i:I.\ \varphi(i,j) \Rightarrow \psi(j)}$$

Classes of fibrations

The Beck-Chevalley condition for existential fibrations:

$$\begin{tikzcd} \mathcal{E}_{I\times J} \arrow[r, "(id_I\times u)^*"] \arrow[d, "\exists_{I,J}"'] & \mathcal{E}_{I\times K} \arrow[d, "\exists_{I,K}"] \arrow[dl, phantom, "\swarrow"] \\ \mathcal{E}_J \arrow[r, "u^*"'] & \mathcal{E}_K \end{tikzcd}$$

given via the mate construction is an isomorphism.

The Frobenius reciprocity for existential fibrations:

$$\begin{tikzcd} \mathcal{E}_{I\times J} \times \mathcal{E}_J \arrow[r, "id\times \pi_J^*"] \arrow[d, "\exists_{I,J}\times id"'] & \mathcal{E}_{I\times J} \times \mathcal{E}_{I\times J} \arrow[r, "\wedge"] \arrow[d, phantom, "\swarrow"] & \mathcal{E}_{I\times J} \arrow[d, "\exists_{I,J}"] \\ \mathcal{E}_J \times \mathcal{E}_J \arrow[rr, "\wedge"'] & & \mathcal{E}_J \end{tikzcd}$$

given via the mate construction is an isomorphism.

Example: For an OFS (E, M) on a finitely complete category $\mathcal{C}$

$\mathcal{C}^{\rightarrow} \supseteq M \xrightarrow{cod} \mathcal{C}$ is a primary fibration.

If (E, M) is stable, then it is elementary and existential.

Classes of fibrations

$$\begin{tikzcd} \mathcal{E}_J \arrow[r, shift right, "\pi_J^*"'] \arrow[r, phantom, "\perp"] & \mathcal{E}_{I\times J} \arrow[l, shift right, "\exists_{I,J}"'] \end{tikzcd} \quad ; \quad \exists_{I,J}\big(\varphi(i:I,\, j:J)\big) \text{ “}\equiv\text{” } \exists i:I.\ \varphi(i,j)$$

The BC condition : $\exists i:I.\ \varphi(i, u(k)) \equiv \big(\exists i:I.\ \varphi(i,j)\big)[u(k)/j]$

The Frobenius reciprocity : $\exists i:I.\big(\varphi(i,k) \wedge \psi(k)\big) \equiv \big(\exists i:I.\ \varphi(i,k)\big) \wedge \psi(k)$

Example : For an OFS (E, M) on a finitely complete category $\mathcal{C}$

$\mathcal{C}^{\to} \supseteq \mathcal{M} \xrightarrow{\mathrm{cod}} \mathcal{C}$ is a primary fibration.

If (E, M) is stable, then it is existential.

Virtual double categories

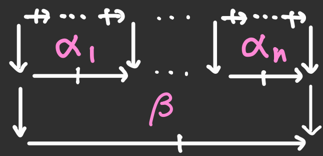
A virtual double category (VDC) $\mathbb{D}$ consists of

- objects
- vertical arrows
- horizontal arrows "without composition"
- virtual cells

$$\begin{tikzcd} A_0 \arrow[r, "R_1", "|" marking] \arrow[d, "f_0"'] & A_1 \arrow[r, "|" marking] \arrow[rr, phantom, ""] & \cdots \arrow[r, "R_n", "|" marking] & A_n \arrow[d, "f_1"] \\ B_0 \arrow[rrr, "S"', "|" marking] \arrow[urrr, phantom, "\alpha"] & & & B_1 \end{tikzcd}$$

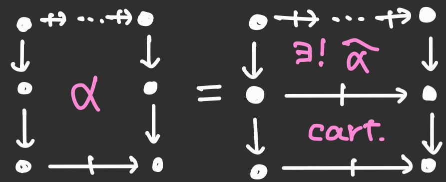
← paths of finite length

with composition

$\beta\{\alpha_1, \dots, \alpha_n\}$:= 

.

Restrictions and cartesianness are defined similarly to those for double categories.



A VDC $\mathbb{D}$ is called fibrant (FVDC) if any niche has its restriction.

From fibrations to VDCs

Construction

$$\begin{array}{c} \mathcal{E} \\ p \big\downarrow \\ \mathcal{B} \end{array} : \text{a primary fibration} \rightsquigarrow \text{a cartesian fibrant VDC } \mathbb{V}(p)$$

- The vertical part of $\mathbb{V}(p)$ is $\mathcal{B}$.
- The horizontal arrows $A \overset{R}{\to} B$ are the objects $R \in \mathcal{E}_{A \times B}$.
- The virtual cells

$$\begin{tikzcd} A_0 \arrow[r, "R_1"] \arrow[d, "f_0"'] & A_1 \arrow[r, "R_2"] \arrow[d, phantom, "\alpha"] & A_2 \arrow[d, "f_1"] \\ B_0 \arrow[rr, "S"'] & & B_1 \end{tikzcd}$$

are the arrows

$\alpha : \pi_{01}^* R_1 \wedge \pi_{12}^* R_2 \longrightarrow S$ in $\mathcal{E}$ above $A_0 \times A_1 \times A_2 \xrightarrow{\pi_{02}} A_0 \times A_1 \xrightarrow{f_0 \times f_1} B_0 \times B_1$ in $\mathcal{B}$.

Thm $\mathbb{V}$ gives a 2-functor $\mathbf{PrimFib} \longrightarrow \mathbf{FibVDbl}_{\mathbf{cart}}$

Example

An OFS (E, M) on a category with finite limits $\mathcal{C}$ gives a primary fibration $\begin{array}{c} M \\ \downarrow \scriptstyle{cod} \\ \mathcal{C} \end{array}$.

$\mathbb{V}\left(\begin{array}{c} M \\ \downarrow \scriptstyle{cod} \\ \mathcal{C} \end{array}\right)$	$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$	$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B$	$\begin{tikzcd} A_0 \arrow[r, "R_1", "\shortmid" marking] \arrow[d, "f_0"'] & A_1 \arrow[r, "R_2", "\shortmid" marking] \arrow[d, phantom, "\alpha"] & A_2 \arrow[d, "f_1"] \\ B_0 \arrow[rr, "S"', "\shortmid" marking] & & B_1 \end{tikzcd}$
	$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$ in $\mathcal{C}$	$\begin{tikzcd} R \arrow[d, tail, "\langle l_R, r_R \rangle"'] \\ A \times B \end{tikzcd}$ in M	$\begin{tikzcd} \pi_{01}^* R_1 \wedge \pi_{12}^* R_2 \arrow[r, tail] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & A_0 \times A_1 \times A_2 \arrow[d, "\langle f_0, f_1 \rangle"] \\ S \arrow[r, tail] & B_0 \times B_1 \end{tikzcd}$

By internal logic, α is interpreted as a Horn sequence :

$$\boxed{x_0 : A_0,\ x_1 : A_1,\ x_2 : A_2 \mid R_1(x_0, x_1) \wedge R_2(x_1, x_2) \Rightarrow S(f_0(x_0), f_1(x_1))}$$

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$\exists$ in VDCs

A path of horizontal arrows $A_0 \overset{R_1}{\nrightarrow} A_1 \nrightarrow \cdots \overset{R_n}{\nrightarrow} A_n$ is composable if it comes equipped with a horizontal arrow $A_0 \overset{\odot R}{\nrightarrow} A_n$ and a cell

$$\begin{tikzcd} A_0 \arrow[r, "R_1", "|"{marking}] \arrow[d, equal] & A_1 \arrow[r, "|"{marking}] & \cdots \arrow[r, "R_n", "|"{marking}] & A_n \arrow[d, equal] \\ A_0 \arrow[rrr, "\odot R"', "|"{marking}] \arrow[rrr, phantom, "\gamma_{R_1,\dots,R_n}", shift left=4] & & & A_n \end{tikzcd}$$

s.t.

$$\begin{tikzcd} C \arrow[r, "|"{marking}] \arrow[d] & \cdots \arrow[r, "|"{marking}] & A_0 \arrow[r, "R_1", "|"{marking}] & \cdots \arrow[r, "R_n", "|"{marking}] & A_n \arrow[r, "|"{marking}] & \cdots \arrow[r, "|"{marking}] & D \arrow[d] \\ E \arrow[rrrrrr, "S"', "|"{marking}] \arrow[rrrrrr, phantom, "\alpha", shift left=4] & & & & & & B \end{tikzcd}
=
\begin{tikzcd} C \arrow[r, "|"{marking}] \arrow[d, equal] & \cdots \arrow[r, "|"{marking}] & A_0 \arrow[r, "R_1", "|"{marking}] \arrow[d, equal] & \cdots \arrow[r, "R_n", "|"{marking}] & A_n \arrow[r, "|"{marking}] \arrow[d, equal] & \cdots \arrow[r, "|"{marking}] & D \arrow[d, equal] \\ C \arrow[r, "|"{marking}] \arrow[d] & \cdots \arrow[r, "|"{marking}] & A_0 \arrow[rr, "\odot R"', "|"{marking}] \arrow[rr, phantom, "\gamma", shift left=4] & & A_n \arrow[r, "|"{marking}] & \cdots \arrow[r, "|"{marking}] & D \arrow[d] \\ E \arrow[rrrrrr, "S"', "|"{marking}] \arrow[rrrrrr, phantom, "\exists!\,\tilde{\alpha}", shift left=4] & & & & & & B \end{tikzcd}$$

This is a certain kind of double colimits.

∃ in VDCs

The universal property of the composite $R_1 \odot R_2$ can be seen as :

$$\begin{tikzcd}
Z \arrow[r, "P", "\shortmid"{marking}] \arrow[d, equal] & X_0 \arrow[r, "R_1", "\shortmid"{marking}] & X_1 \arrow[r, "R_2", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & X_2 \arrow[r, "Q", "\shortmid"{marking}] & Y \arrow[d, equal] \\
Z \arrow[rrrr, "S"', "\shortmid"{marking}] & & & & Y
\end{tikzcd}
\quad \overset{1:1}{\longleftrightarrow} \quad
\begin{tikzcd}
Z \arrow[r, "P", "\shortmid"{marking}] \arrow[d, equal] & X_0 \arrow[r, "R_1 \odot R_2", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & X_2 \arrow[r, "Q", "\shortmid"{marking}] & Y \arrow[d, equal] \\
Z \arrow[rrr, "S"', "\shortmid"{marking}] & & & Y
\end{tikzcd}$$

The logical interpretation is $\exists x_1.\, R_1(\bullet, x_1) \wedge R_2(x_1, \bullet)$.

$$\begin{array}{c}
P(z, x_0),\ R_1(x_0, x_1),\ R_2(x_1, x_2),\ Q(x_2, y) \Longrightarrow S(z, y) \\
\hline\hline
P(z, x_0),\ \exists x_1.\big(R_1(x_0, x_1) \wedge R_2(x_1, x_2)\big),\ Q(x_2, y) \Longrightarrow S(z, y)
\end{array}$$

The composability of paths of positive length is

the double categorical counterpart of the existential quantifier.

$\exists$ in VDCs

Thm p is existential iff $\mathbb{V}(p)$ is composable.

$$\begin{tikzcd} \mathrm{ExFib} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{CFibVDbl}_{\mathrm{cart}} \arrow[d] \\ \mathrm{PrimFib} \arrow[r, "\mathbb{V}"'] & \mathrm{FibVDbl}_{\mathrm{cart}} \end{tikzcd}$$

is a 2-pullback.

Logic	Fibrations	FVDCs
$\exists$	existential fibrations	cartesian composable FVDCs

∃ in VDCs

Thm p is existential iff $\mathbb{V}(p)$ is composable.

$$
\begin{tikzcd}
\mathcal{E}\mathrm{xFib} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{CFibVDbl}_{\mathrm{cart}} \arrow[d] \\
\mathrm{PrimFib} \arrow[r, "\mathbb{V}"'] & \mathrm{FibVDbl}_{\mathrm{cart}}
\end{tikzcd}
$$

is a 2-pullback.

Logic	Fibrations	FVDCs
∃	existential fibrations	cartesian composable FVDCs
=	elementary fibrations	cartesian unital FVDCs

VDCs vs hyperdoctrines (Future work)

In terms of VDCs,

- some constructions using relational structures can be well handled. (e.g., quotient completions)
- quantifiers $\exists$, $\forall$ and equality $=$ are captured by double (co) limits, which are in the scope of formal category theory.

The core problem

It seems that the internal logic of VDCs is a proper extension of regular logic.

$$\exists x. (\, p(x) \wedge q(x) \,)$$

$$1 \xrightarrow[\quad\shortmid\quad]{p(x)} A \xrightarrow[\quad\shortmid\quad]{q(x)} 1 \;\neq\; 1 \xrightarrow[\quad\shortmid\quad]{\top} A \xrightarrow[\quad\shortmid\quad]{p(x) \wedge q(x)} 1$$

Might be called directed logic?

Thank you!

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$=$ in VDC

- A primary fibration is called elementary if the reindexings $\mathcal{E}_{I\times I\times J} \xrightarrow{(\Delta_I \times id_J)^*} \mathcal{E}_{I\times J}$ for all I and J have left adjoints and they satisfy the Beck-Chevalley condition and the Frobenius reciprocity.

- The composite of a path of length 0 is called a unit.

- The universal property of the unit U_X is seen as :

$$
\begin{tikzcd}
Z \arrow[r, "P", "\shortmid"{marking}] \arrow[d, equal] & X \arrow[r, "Q", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & Y \arrow[d, equal] \\
Z \arrow[rr, "S"', "\shortmid"{marking}] & & Y
\end{tikzcd}
\quad \xleftrightarrow{1:1} \quad
\begin{tikzcd}
Z \arrow[r, "P", "\shortmid"{marking}] \arrow[d, equal] & X \arrow[r, "U_X", "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow"] & X \arrow[r, "Q", "\shortmid"{marking}] & Y \arrow[d, equal] \\
Z \arrow[rrr, "S"', "\shortmid"{marking}] & & & Y
\end{tikzcd}
$$

$$
\frac{P(z,x),\ Q(x,y) \Rightarrow S(z,y)}{P(z,x),\ x = x',\ Q(x',y) \Rightarrow S(z,y)}
$$

- $$\begin{tikzcd} \mathsf{ElemFib} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{UFibVDbl}_{\mathrm{cart}} \arrow[d] \\ \mathsf{PrimFib} \arrow[r, "\mathbb{V}"'] & \mathsf{FibVDbl}_{\mathrm{cart}} \end{tikzcd}$$ is a 2-pullback.

Combining this with the 2-pullback in the previous slide,

we have $\mathsf{ElemExFib} \xrightarrow{\mathbb{V}} \mathsf{FibDbl}_{\mathrm{cart}}$.

An equivalence

Restricting this to the full sub-2-categories, we have an equivalence :

$$\mathrm{ElemExFib}_{\mathrm{Frob}} \xrightarrow{\simeq} \mathcal{E}q_{\mathrm{cart},\mathrm{Frob}}$$

$\mathrm{ElemExFib}_{\mathrm{Frob}}$ is the 2-category of fibrations satisfying the BC

for all the pullback of the form

$$\begin{tikzcd}
 & I \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
I^m \arrow[dr] & & I^n \arrow[dl] \\
 & I^\ell &
\end{tikzcd}$$

where

$$\begin{tikzcd}
 & 1 \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
m \arrow[ur] & & n \arrow[ul] \\
 & \ell \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

in **FinSet**.