# The regular completion of $\mathbf{Cat}$ and its relation to lazy categories

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# Introduction

- The theory of lazy categories aims to fix quotients in $\mathbf{Cat}$ by embedding it in a nice topos (Verity)
  - What's wrong with quotients in $\mathbf{Cat}$?
  - Perhaps the main thing is that they are not stable under pullback
  - But they just seem wrong – unruly and just plain nasty
  - Exercise: Describe the monoid of endomorphisms $\mathbf{X}(2,2)$ for the coequalizer

$$\mathbf{Set_0} \underset{3\times(\ )}{\overset{2\times(\ )}{\rightrightarrows}} \mathbf{Set_0} \twoheadrightarrow \mathbf{X}$$

- Another way of "fixing quotients" is the regular completion of Carboni

- We compare the two

# Regular categories

- Regular categories are categories with a good factorization system
  - finite limits
  - every morphism factors as a regular epi followed by a mono
  - regular epis are stable under pullback
- A regular epimorphism is a coequalizer of some pair of morphisms
- They are orthogonal to monomorphisms

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & C \arrow[d, tail] \\
B \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}
$$

**Examples**
- Algebraic categories
- Any topos

- But not **Cat**, ex:

$$
\begin{tikzcd}
 & 1 \quad 1' \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] & & 2
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
 & 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] & & 2
\end{tikzcd}
$$

does not factor

# Regular completion

- A regular functor is a functor that preserves finite limits (i.e. is left exact) and regular epimorphisms
- We have a forgetful 2-functor

$$U\colon \mathscr{R}eg \longrightarrow \mathscr{L}ex$$

and we want a left (2−)adjoint for it
- Idea: Formally add images

$$\begin{tikzcd} A \arrow[dd, "f"'] \arrow[dr] & & & C \arrow[dl] \arrow[dd, "g"] \\ & \mathrm{Im}(f) \arrow[r, dashed, "?"] \arrow[dl] & \mathrm{Im}(g) \arrow[dr, tail] & \\ B & & & D \end{tikzcd}$$

# The construction (Carboni *et al.*)

$\mathbf{A}$ with finite limits

Reg($\mathbf{A}$)

- objects: morphisms of $\mathbf{A}$
- morphism: $h\colon (f) \longrightarrow (g)$

$$\begin{tikzcd} A \arrow[r, dashed, "\overline{h}"] \arrow[d, "f"'] \arrow[dr, "h"] & C \arrow[d, "g"] \\ B & D \end{tikzcd}$$

$h$ lifts to some $\overline{h}$

coequalizes the kernel pair of $f$

- Terminal object is $1\colon 1 \longrightarrow 1$
- Pulbacks = pullbacks of $h$'s
- Factorization

$$\begin{tikzcd} A \arrow[r, dashed, "1"] \arrow[d, "f"'] \arrow[dr, "h"] & A \arrow[r, dashed, "\overline{h}"] \arrow[d, "h"] \arrow[dr, "h"] & C \arrow[d, "g"] \\ B & D & D \end{tikzcd}$$

# Universal property

$$\begin{tikzcd} & \mathrm{Reg}(\mathbf{A}) \arrow[dd, dashed, "\exists \Psi \mathrm{reg}"] \\ \mathbf{A} \arrow[ur, tail] \arrow[dr, "\forall \mathrm{lex}\ \Phi"'] & \\ & \mathbf{B}\ \mathrm{reg} \end{tikzcd}$$

### Theorem

$\mathrm{Reg}$ *is a (pseudo) left adjoint to* $U\colon \mathscr{R}eg \longrightarrow \mathscr{L}ex$*, and* $U$ *is* $2$*-monadic.*

- $\Psi(f) = \mathrm{Im}(\Phi(f))$.
- We can apply $\mathrm{Reg}$ to $\mathbf{Cat}$ (size considerations notwithstanding) and get an extension of $\mathbf{Cat}$

  $$\mathbf{Cat} \rightarrowtail \mathrm{Reg}(\mathbf{Cat})$$

  where images (and relations) work well.
- But there's another way – the lazy way!

# The topos hull of $\mathbf{Cat}$

- Embed $\mathbf{Cat}$ into the most fitting Grothendieck topos – sheaves on the canonical topology
- The covering families $\langle F_i \colon \mathbf{A}_i \longrightarrow \mathbf{B} \rangle$ are those for which the induced functor

$$F \colon \Sigma \mathbf{A}_i \twoheadrightarrow \mathbf{B}$$

  is onto on composable pairs
- The $F \colon \mathbf{A} \longrightarrow \mathbf{B}$ that are onto on pairs are precisely the stable regular epis in $\mathbf{Cat}$: we call them *covers*

- $\mathrm{Sh}(\mathbf{Cat}, \mathrm{Cov}) \simeq \mathbf{Set}^{\Delta_2^{op}}$

# Lazy categories

### Definition

A *lazy category* is a diagram of sets and functions

$$\begin{tikzcd} A_2 \arrow[r, shift left=6] \arrow[r] \arrow[r, shift right=6] & A_1 \arrow[l, shift right=3, "\lambda"'] \arrow[l, shift left=3, "\rho"] \arrow[r, shift left=4] \arrow[r, shift right=4] & A_0 \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

satisfying well-known equations.

A *lazy functor* is given by three functions commuting with the structure maps.

- We have a full embedding $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$
  - $\mathbf{LCat}$ is a Grothendieck topos, so it's like $\mathbf{Set}$
  - But it's also like $\mathbf{Cat}$

# The device

- Introduce category like notation
  - Elements of $A_0$ are called *objects*, denoted $A, B, C, \ldots$
  - Elements of $A_1$ are called *morphisms*, denoted $f\colon A \longrightarrow B, \ldots$
  - Elements of $A_2$ are called *composers*, denoted

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

  - Identities $1_A\colon A \longrightarrow A$ and

$$\begin{tikzcd} & B \arrow[dr, "1_B"] \arrow[d, phantom, "\lambda_f"] & \\ A \arrow[ur, "f"] \arrow[rr, "f"'] & {} & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "f"] \arrow[d, phantom, "\rho_f"] & \\ A \arrow[ur, "1_A"] \arrow[rr, "f"'] & {} & B \end{tikzcd} \qquad \lambda_{1_A} = \rho_{1_A} = \iota_A$$

- Functors preserve all of the above

# Example

Let $\mathbf{A}$ be a category and define a lazy category $\mathrm{Sub}(\mathbf{A})$ as follows

- Objects are subcategories $\mathbf{B} \rightarrowtail \mathbf{A}$
- Morphisms $M\colon \mathbf{B} \to \mathbf{C}$ are subsets $M \subseteq \mathrm{Mor}(\mathbf{A})$ whose elements have domain in $\mathbf{B}$ and codomain in $\mathbf{C}$ and closed under composition by morphisms in $\mathbf{B}$ and $\mathbf{C}$ (ideal)
- Composers

$$\begin{tikzcd} & \mathbf{C} \arrow[dr, "N"] & \\ \mathbf{B} \arrow[ur, "M"] \arrow[rr, "P"'] & & \mathbf{D} \end{tikzcd}$$

  containments $NM \subseteq P$

- For any category $\mathbf{X}$ we have a bijection

$$\frac{\mathbf{X} \longrightarrow \mathrm{Sub}(\mathbf{A}) \text{ in } \mathbf{LCat}}{\mathbf{B} \rightarrowtail \mathbf{A} \times \mathbf{X} \text{ in } \mathbf{Cat}}$$

# Properties

## Theorem

(1) $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *has a left adjoint* $\mathrm{L}$

(2) $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *preserves all limits*

(3) $\mathbf{Cat} \rightarrowtail \mathbf{LCat}$ *preserves coproducts and covers*

(4) $\mathbf{Cat}$ *is an exponential ideal in* $\mathbf{LCat}$

(5) *Every lazy category has a cover by a projective category*

Concerning (1) – For a lazy category $\mathbf{A}$, $L\mathbf{A}$ is obtained by taking the free category on the morphisms of $\mathbf{A}$, i.e. paths, and modding out by the congruence generated by identifying

$$A \xrightarrow{f} B \xrightarrow{g} C \quad \text{with} \quad A \xrightarrow{h} C$$

when there is a composer

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=4, "\alpha"] & & C \end{tikzcd}$$

Concerning (5) – The projectives in $\mathbf{LCat}$ are sums of $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$

$$\mathbf{LCat}(\mathbf{3},\mathbf{A}) \bullet \mathbf{3} \twoheadrightarrow \mathbf{A}$$

# $\mathbf{LCat}$ vs $\mathrm{Reg}(\mathbf{Cat})$

$$
\begin{tikzcd}
 & \mathrm{Reg}(\mathbf{Cat}) \arrow[d, dashed, "\Psi"] \\
\mathbf{Cat} \arrow[ur, tail] \arrow[r, tail] & \mathbf{LCat}
\end{tikzcd}
$$

$$\Psi(F) = \mathrm{Im}(F)$$

$$\mathbf{A} \twoheadrightarrow \mathrm{Im}(F) \rightarrowtail \mathbf{B}$$

## Theorem

(1) *$\Psi$ has a left adjoint $\Lambda$*

(2) *The induced monad $\Psi\Lambda$ is idempotent*

(3) *The category of algebras (= fixed points) of $\Psi\Lambda$ is the full subcategory of $\mathrm{L}\mathbf{Cat}$ determined by those lazy categories embeddable in a category*

Concerning (1)

$$\Lambda(\mathbf{A}) = (\mathbf{LCat}(\mathbf{3}, \mathbf{A}) \bullet \mathbf{3} \twoheadrightarrow \mathbf{A} \rightarrow L\mathbf{A})$$

# Partial categories

- A lazy subcategory of an actual category is called a *partial category*
- Every partial category is $\Psi$ of an object of $\mathrm{Reg}(\mathbf{Cat})$

$$\mathbf{LCat}(\mathbf{3},\mathbf{A}) \bullet \mathbf{3} \twoheadrightarrow \mathbf{A} \rightarrowtail \mathbf{B}$$

- Every functor between partial categories is $\Psi$ of a morphism in $\mathrm{Reg}(\mathbf{Cat})$
- However $\Psi$ is not full (it is faithful)

# Properties of $\mathbf{ParCat}$

Theorem

(1) *The inclusion* $\mathbf{ParCat} \rightarrowtail \mathbf{LCat}$ *has a left adjoint given by the image of* $\eta\mathbf{A}\colon \mathbf{A} \longrightarrow \mathrm{L}\mathbf{A}$

$$\mathbf{A} \twoheadrightarrow \mathrm{Im}(\eta\mathbf{A}) \rightarrowtail \mathrm{L}\mathbf{A}$$

(2) $\mathbf{ParCat}$ *is an exponential ideal in* $\mathbf{LCat}$

(3) $\mathbf{ParCat}$ *is regular*

(4) $\mathbf{ParCat}$ *is not locally cartesian closed*

## Characteristics of partial categories

- Partial categories are *thin*, in fact *gaunt*

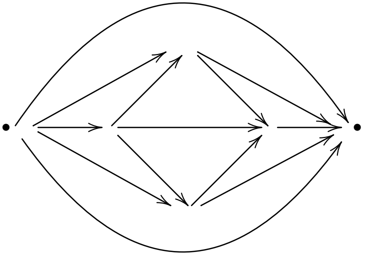
  For any morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ there is at most one composer

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}$$

- Partial categories are weakly associative: if $(fg)h$ and $f(gh)$ are both defined then they are equal. This holds for any two brackettings of an $n$-composite

$$f_1 \; f_2 \; \ldots \; f_n$$

- Not sufficient



# Weird associativity?

- $\mathbf{A}$ is a partial category if and only if $\eta\mathbf{A}\colon \mathbf{A} \longrightarrow \mathrm{L}\mathbf{A}$ is monic
- $x \sim y$

$$
\begin{tikzcd}
{} \arrow[rrrr, "x"] \arrow[d, equal] & & {} \arrow[d, phantom, "\alpha"] & & {} \arrow[d, equal] \\
{} \arrow[rr] \arrow[d, equal] & & {} \arrow[rr] \arrow[d, equal] & {} \arrow[d, phantom, "\alpha"] & {} \arrow[d, equal] \\
{} \arrow[rr] \arrow[d, equal] & {} \arrow[d, phantom, "\alpha"] & {} \arrow[r] \arrow[d, equal] & {} \arrow[r] \arrow[d, equal] & {} \arrow[d, equal] \\
{} \arrow[r] \arrow[d, equal] & {} \arrow[r] \arrow[d, equal] & {} \arrow[r] \arrow[d, phantom, "\alpha"] & {} \arrow[r] \arrow[d, equal] & {} \arrow[d, equal] \\
{} \arrow[r] \arrow[d, equal] & {} \arrow[rr] \arrow[d, equal] & {} \arrow[d, phantom, "\alpha"] & {} \arrow[r] \arrow[d, equal] & {} \arrow[d, equal] \\
{} \arrow[r] \arrow[d, equal] & {} \arrow[r] \arrow[d, phantom, "\alpha"] & {} \arrow[r] \arrow[d, equal] & {} \arrow[r] \arrow[d, equal] & {} \arrow[d, equal] \\
{} \arrow[rr] \arrow[d, equal] & & {} \arrow[r] \arrow[d, equal] & {} \arrow[r] \arrow[d, phantom, "\alpha"] & {} \arrow[d, equal] \\
{} \arrow[rr] \arrow[d, equal] & & {} \arrow[rr] \arrow[d, phantom, "\alpha"] & & {} \arrow[d, equal] \\
{} \arrow[rrrr, "y"'] & & & & {}
\end{tikzcd}
$$

- **Question:** Can this be phrased as some kind of associativity?

# Conclusion

- While $\mathrm{Reg}(\mathbf{Cat})$ is an interesting category is not a good substitute for $\mathbf{Cat}$
  - Kills the good quotients
  - Unwieldy
  - Not even well-powered (Johnstone), so not locally presentable nor a topos

- Its image in $\mathbf{LCat}$ is $\mathbf{ParCat}$
  - Many interesting properties
  - Easy to work with
  - But it's not a topos (nor a quasi-topos)
  - Keep in mind ... but $\mathbf{LCat}$ rules!

Thank you,
and come again!