Towards Negation in Double-Categorical Logic

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Big Idea

Typed First-Order Logic $\longleftrightarrow$ (Virtual) Double Categories

Big Idea

Typed First-Order Logic <-----------> (Virtual) Double Categories

Hyperdoctrines

Fibrations

Big Idea

$$
\begin{tikzcd}
\text{Typed First-Order Logic} \arrow[rr, leftrightarrow, dashed] \arrow[dr, leftrightarrow, "\text{Hyperdoctrines}"'] & & \text{(Virtual) Double Categories} \arrow[dl, leftrightarrow, "\text{Bil/uni constructions}"] \\
& \text{Fibrations} &
\end{tikzcd}
$$

Big Idea

$$
\begin{tikzcd}
\text{Typed First-Order Logic} \arrow[rr, <->, dashed] \arrow[dr, dash, line width=2pt, "\text{Hyperdoctrines}"'] & & \text{(Virtual) Double Categories} \arrow[dl, <->, "\text{Bi/uni constructions}"] \\
& \text{Fibrations} &
\end{tikzcd}
$$

(Typed) First-Order Logic

Definition A first-order specification $(\Sigma, \Pi, \mathcal{A})$ consists of:

- a pair $\Sigma = (T, \mathcal{F})$ where T is a set of basic types and $\mathcal{F}$ is a mapping $T^* \times T \to \mathrm{Set}$ which assigns a set of function symbols to a finite sequence of input types and an output type;
- a mapping $\Pi : T^* \to \mathrm{Set}$ which assigns a set of predicate symbols to each finite sequence of basic types;
- a collection $\mathcal{A}$ of sequents in the first-order language of (Σ, Π), called axioms.

Sequents consist of:

$$\underbrace{v_1 : \sigma_1, v_2 : \sigma_2, \ldots, v_n : \sigma_n}_{\text{Type context}} \mid \overbrace{\varphi_1, \varphi_2, \ldots, \varphi_m}^{\text{Proposition context}} \vdash \overbrace{\psi}^{\text{Consequent}}$$

Language of First-Order Logic

Atomic propositions:

$$M = M'$$

for M, M' terms of type σ

$$P(M_1, M_2, \dots, M_n)$$

for $P : \sigma_1, \dots, \sigma_n$ a predicate symbol and $M_i : \sigma_i$, $1 \le i \le n$, appropriately typed terms

New propositions can be built using connectives and quantifiers

$\top$: truth, universally true proposition

$\bot$: falsum, universally false proposition

$\varphi \wedge \psi$: φ and ψ

$\varphi \vee \psi$: φ or ψ

$\exists x{:}\sigma.\, \varphi$: there exists x of type σ such that φ

$\forall x{:}\sigma.\, \varphi$: for all x of type σ, φ

$\varphi \Rightarrow \psi$: if φ then ψ

$\neg\varphi$: not φ

that obey natural deduction rules.

Language of First-Order Logic

Atomic propositions:

$$M = M'$$

for M, M' terms of type σ

$$P(M_1, M_2, \ldots, M_n)$$

for $P : \sigma_1, \ldots, \sigma_n$ a predicate symbol and $M_i : \sigma_i$, $1 \leq i \leq n$, appropriately typed terms

New propositions can be built using connectives and quantifiers

Regular logic

$\top$: truth, universally true proposition

$\varphi \wedge \psi$: φ and ψ

$\exists x : \sigma.\ \varphi$: there exists x of type σ such that φ

$\varphi \Rightarrow \psi$: if φ then ψ

$\bot$: falsum, universally false proposition

$\varphi \vee \psi$: φ or ψ

$\forall x : \sigma.\ \varphi$: for all x of type σ, φ

$\neg \varphi$: not φ

that obey natural deduction rules.

Language of First-Order Logic

Atomic propositions:

$$M = M'$$

for M, M' terms of type σ

$$P(M_1, M_2, \ldots, M_n)$$

for $P : \sigma_1, \ldots, \sigma_n$ a predicate symbol and $M_i : \sigma_i$, $1 \leq i \leq n$, appropriately typed terms

New propositions can be built using connectives and quantifiers

Coherent logic

$\top$: truth, universally true proposition

$\bot$: falsum, universally false proposition

$\varphi \wedge \psi$: φ and ψ

$\varphi \vee \psi$: φ or ψ

$\exists x : \sigma .\, \varphi$: there exists x of type σ such that φ

$\forall x : \sigma .\, \varphi$: for all x of type σ, φ

$\varphi \Rightarrow \psi$: if φ then ψ

$\neg \varphi$: not φ

that obey natural deduction rules.

Fibrations

Definition Given a functor $p : \mathcal{E} \to \mathcal{B}$, a morphism $f : x \to y$ in $\mathcal{E}$ is called <u>cartesian</u> if for any morphism $g : z \to y$ in $\mathcal{E}$ and $w : p(z) \to p(x)$ in $\mathcal{B}$ such that $p(f) \circ w = p(g)$, there exists a unique $\hat{w} : z \to x$ such that $g = f \circ \hat{w}$ and $p(\hat{w}) = w$:

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "p"'] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[dr, dashed, "\exists! \hat{w}"'] \arrow[drr, bend left, "\forall g"] & & \\
& x \arrow[r, "f"', "\text{cartesian}"'] & y
\end{tikzcd}$$

$$\begin{tikzcd}
p(z) \arrow[dr, "\forall w"'] \arrow[drr, bend left, "p(g)"] & & \\
& p(x) \arrow[r, "p(f)"'] & p(y)
\end{tikzcd}$$

Definition A functor $p : \mathcal{E} \to \mathcal{B}$ is a <u>fibration</u> if for all objects $y \in \mathcal{E}$ and $f_0 : x_0 \to p(y)$, there is a cartesian morphism $f : x \to y$ such that $p(f) = f_0$.

Fibrations (cont'd)

Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration. A choice of cartesian lifting for every y and f_0 is called a *cleavage*, and a fibration together with a cleavage is called *cloven*.

Fact In a cloven fibration $p: \mathcal{E} \to \mathcal{B}$, every morphism $f: A \to B$ in $\mathcal{B}$ induces a *reindexing functor* $f^*: \mathcal{E}_B \to \mathcal{E}_A$ between the fibres:

Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration. We say that p has *fibred $\diamond$'s* (where $\diamond$ is some categorical property or structure) if all fibre categories have $\diamond$'s and all reindexing functors preserve $\diamond$'s.

Special Conditions

Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration, and
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$
a commuting square in $\mathcal{B}$ s.t. the reindexing functors $f^*: \mathcal{E}_C \to \mathcal{E}_A$ and $g^*: \mathcal{E}_D \to \mathcal{E}_B$ have left adjoints $\exists_f$ and $\exists_g$, respectively. This square is said to satisfy the Beck-Chevalley condition if the canonical natural transformation $\exists_f h^* \Longrightarrow k^* \exists_g$ is an isomorphism.

Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with fibred finite products, and $f: A \to B$ a morphism in $\mathcal{B}$ such that f^* has a left adjoint $\exists_f$. We say that f satisfies Frobenius reciprocity if

$$\exists_f(\varphi \wedge f^*\psi) \cong (\exists_f \varphi) \wedge \psi$$

for all $\varphi \in \mathcal{E}_A$ and $\psi \in \mathcal{E}_B$.

Fibrations of Contexts

Definition Let $(\Sigma, \Pi, \mathcal{A})$ be a first-order specification. This gives rise to a <u>fibration of contexts</u> $\mathcal{L}(\Sigma, \Pi, \mathcal{A}) \longrightarrow \mathcal{Cl}(\Sigma)$ as follows:

- objects of $\mathcal{Cl}(\Sigma)$ are contexts, i.e. finite sequences of variable declarations, written as
$$\Gamma = (v_1 : \sigma_1, v_2 : \sigma_2, \dots, v_n : \sigma_n)$$

- morphisms of $\mathcal{Cl}(\Sigma)$ $\Gamma \to \Delta$, where $\Delta = (v_1 : \tau_1, \dots, v_m : \tau_m)$, are m-tuples of terms $(M_1, \dots, M_m)$ for which we can derive $\Gamma \vdash M_i : \tau_i$ for all $1 \leq i \leq m$.

- objects of $\mathcal{L}(\Sigma, \Pi, \mathcal{A})$ are pairs $\Gamma | \Theta$ consisting of a type context Γ and a proposition context Θ such that all free variables used in Θ are declared in Γ

- morphisms of $\mathcal{L}(\Sigma, \Pi, \mathcal{A})$ $(\Gamma | \Theta) \to (\Gamma' | \Theta')$ are context morphisms $\vec{M} : \Gamma \to \Gamma'$ in $\mathcal{Cl}(\Sigma)$ such that for each proposition ψ in Θ' one can derive $\Gamma | \Theta \vdash \psi[\vec{M}/\vec{v}]$.

Examples

Definition A fibration $p : \mathcal{E} \to \mathbb{B}$ is said to be elementary existential if:

- $\mathbb{B}$ has finite products,
- $\mathcal{E}$ has fibred finite products
- For every parameterized diagonal $\Delta : X \times Y \to X \times Y \times Y$ in $\mathbb{B}$, the reindexing functor $\Delta^* : \mathcal{E}_{X \times Y \times Y} \to \mathcal{E}_{X \times Y}$ has a left adjoint $\exists_\Delta$ satisfying Beck-Chevalley and Frobenius reciprocity;
- For every projection $\pi_Y : X \times Y \to X$ in $\mathbb{B}$, the reindexing functor $\pi_Y^* : \mathcal{E}_X \to \mathcal{E}_{X \times Y}$ has a left adjoint $\exists_{\pi_Y}$ satisfying Beck-Chevalley and Frobenius reciprocity.

Elementary existential fibrations whose fibres are preorders are called regular fibrations by Jacobs because they capture specifications of regular logic.

Examples (cont'd)

Definition An elementary existential fibration with distributive fibred finite coproducts is called a coherent fibration.

Definition A coherent fibration is said to be a first-order fibration if:

- it is a fibred cartesian closed category, and
- for every projection $\pi_Y : X \times Y \to X$ in $\mathcal{B}$, the reindexing functor $\pi_Y^* : \mathcal{E}_X \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_{\pi_Y}$ satisfying Beck-Chevalley.

Definition A Boolean fibration is a fibration $p : \mathcal{E} \to \mathcal{B}$ such that:

- $\mathcal{B}$ has finite limits;
- for every object X of $\mathcal{B}$, the fibre $\mathcal{E}_X$ is a Boolean algebra;
- for every morphism $f : X \to Y$ in $\mathcal{B}$, the reindexing functor $f^* : \mathcal{E}_Y \to \mathcal{E}_X$ is a homomorphism of Boolean algebras with a left and right adjoint satisfying Beck-Chevalley and Frobenius.

"Shortcomings"

- a bit inconvenient; easier to work with a single categorical object rather than two connected ones
- such objects exist (e.g. toposes, bicategories of relations), but lose the ability to talk about terms and propositions as two distinct classes (in the two examples, one is seen as a special case of the other)
- if only there were categories with two different kinds of morphisms...

Double Categories

Definition A (pseudo) double category $\mathbb{D}$ is a pseudocategory internal to Cat:

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "c"] & \mathbb{D}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathbb{D}_0 \arrow[l, "i" description] \end{tikzcd}$$

Explicitly, $\mathbb{D}$ consists of:

(i) objects $A, B, C, D, \ldots$ (objects of $\mathbb{D}_0$);

(ii) tight arrows $f: A \to B$ (morphisms of $\mathbb{D}_0$);

(iii) loose arrows $R: A \to\!\!\!\!\!+ C$ (objects of $\mathbb{D}_1$);

(iv) cells $\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "|" marking] & D \end{tikzcd}$ (morphisms of $\mathbb{D}_1$).

Associativity and unitality for composition in the tight direction are on the nose; in the loose direction, they hold only up to coherent iso.

Double Categories (cont'd)

Examples

- $\mathbb{Q}(K)$ - quintets in a 2-category K
- $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ - sets, functions, spans
- $\mathbb{P}\mathrm{rof}$ - (small) categories, functors, profunctors
- $\mathbb{R}\mathrm{el}^{\circ}$ - sets, functions, relations

Companions + Conjoints

<u>Definition (Grandis-Paré 2004)</u> Let $f: A \to B$ be a tight arrow in a double category $\mathbb{D}$. A loose arrow $f_*: A \nrightarrow B$ is a <u>companion of f</u> if there exist cells α, β (below) such that $\beta \mid \alpha = 1_{f_*}$ and $\frac{\beta}{\alpha} = id_f$.

$$
\begin{tikzcd}
A \arrow[r, "f_*", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\
B \arrow[r, equal, "|" marking] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[d, "f"] \\
A \arrow[r, "f_*"', "|" marking] & B
\end{tikzcd}
$$

<u>Definition (Grandis-Paré 2004)</u> Let $f: A \to B$ be a tight arrow in a double category $\mathbb{D}$. A loose arrow $f^*: B \nrightarrow A$ is a <u>conjoint of f</u> if there exist cells η, ε (below) such that $\varepsilon \mid \eta = 1_{f^*}$ and $\frac{\eta}{\varepsilon} = id_f$.

$$
\begin{tikzcd}
A \arrow[r, equal, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, equal] \\
B \arrow[r, "f^*"', "|" marking] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "f^*", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "f"] \\
B \arrow[r, equal, "|" marking] & B
\end{tikzcd}
$$

Equipments

<u>Definition-Theorem (Shulman 2008)</u> A double category $\mathbb{D}$ is an equipment if any of the following equivalent conditions hold:

(i) $\langle s,t\rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a fibration;

(ii) $\langle s,t\rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is an opfibration;

(iii) $\mathbb{D}$ has all companions and conjoints.

<u>Examples</u>

- $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ (spans with one leg an identity)
- $\mathbb{P}\mathrm{rof}$ (representable profunctors)
- $\mathbb{R}\mathrm{el}^{\circ}$ (graphs/opgraphs of functions)
- $\mathbb{Q}(\mathcal{K})$ has companions, and has conjoints iff every 1-cell is a map (i.e. has a right adjoint)

Equipments (cont'd)

Definition-Proposition An equipment $\mathbb{D}$ is cartesian iff:

- $\mathbb{D}_0$ has finite products;
- for any pair of objects A, B of $\mathbb{D}$, the category $\mathbb{D}(A,B)$ has finite products;
- the double functors $\Delta : \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and $! : \mathbb{D} \to \mathbb{1}$ have right adjoints in $\mathbb{D}\mathrm{bl}$.

Definition Let $\mathbb{D}$ be a cartesian equipment. An object I of $\mathbb{D}$ is called Frobenius if the isomorphism

$$\Delta_{I*} \Delta_I^* \cong (\Delta_I \times \mathrm{id}_I)^* (\mathrm{id}_I \times \Delta_I)_*$$

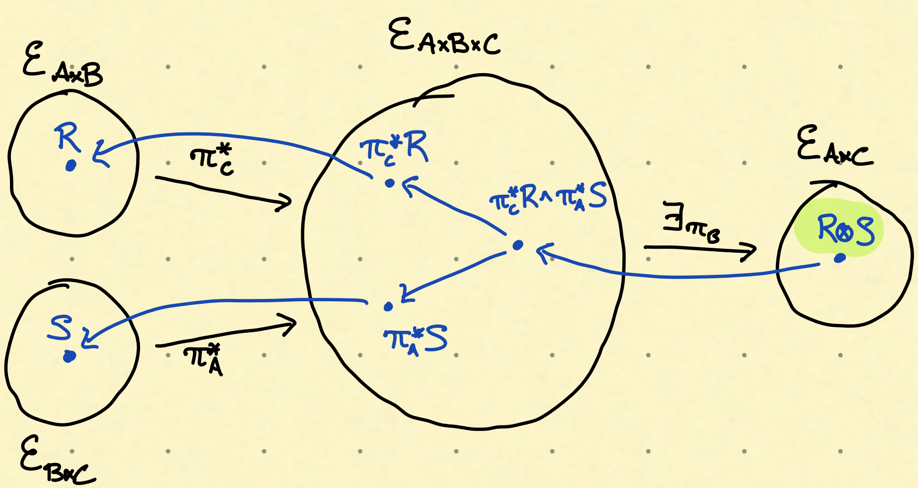
holds. We say that $\mathbb{D}$ is Frobenius if all its objects are Frobenius.

Example $\mathbb{R}\mathrm{el}^\circ$ is a Frobenius cartesian equipment.

$\mathbb{B}\mathrm{il}$ Construction

Definition (Nasu 2025) Let $p : \mathcal{E} \to \mathcal{B}$ be an elementary existential fibration. Then the following data form a Frobenius cartesian equipment $\mathbb{B}\mathrm{il}(p)$:

- the tight part of $\mathbb{B}\mathrm{il}(p)$ is the category $\mathcal{B}$;
- loose arrows $A \nrightarrow B$ in $\mathbb{B}\mathrm{il}(p)$ are objects of $\mathcal{E}$ over $A \times B$;
- cells
$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$$
in $\mathbb{B}\mathrm{il}(p)$ are morphisms $\alpha : R \to S$ in $\mathcal{E}$ over $f \times g$;
- loose identity on A is given by $\exists_{\Delta_A}(\top_A) \in \mathcal{E}_{A\times A}$, where $\top_A$ is the terminal object of $\mathcal{E}_A$;
- given loose arrows $R : A \nrightarrow B$, $S : B \nrightarrow C$, the composite $R \otimes S$ is given as below:



$\mathbb{B}$il Construction (cont'd)

Proof Segment (Associators) Let $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$ be loose arrows in $\mathbb{B}\mathrm{il}(p)$.

$$
\begin{aligned}
\exists_{\pi_C}\left[\pi_D^* \exists_{\pi_B}(\pi_C^* R \wedge \pi_A^* S) \wedge \pi_A^* T\right] &\cong \exists_{\pi_C}\left[\exists_{\pi_B'} \pi_D'^*(\pi_C^* R \wedge \pi_A^* S) \wedge \pi_A^* T\right] && (\text{B-C}) \\
&\cong \exists_{\pi_C}\left[\exists_{\pi_B'}\left(\pi_D'^*(\pi_C^* R \wedge \pi_A^* S) \wedge \pi_B'^* \pi_A^* T\right)\right] && (\text{FR}) \\
&\cong \exists_{\pi_C}\left[\exists_{\pi_B'}\left((\pi_D'^* \pi_C^* R \wedge \pi_D'^* \pi_A^* S) \wedge \pi_B'^* \pi_A^* T\right)\right] && (\pi^* \text{ pres. } \wedge) \\
&\ \ \vdots \\
&\cong \exists_{\pi_B}\left[\exists_{\pi_C'}\left(\pi_C'^* \pi_D^* R \wedge (\pi_A'^* \pi_D^* S \wedge \pi_A'^* \pi_B^* T)\right)\right] \\
&\cong \exists_{\pi_B}\left[\exists_{\pi_C'}\left(\pi_C'^* \pi_D^* R \wedge \pi_A'^*(\pi_D^* S \wedge \pi_B^* T)\right)\right] && (\pi^* \text{ pres. } \wedge) \\
&\cong \exists_{\pi_B}\left[\pi_D^* R \wedge \exists_{\pi_C'} \pi_A'^*(\pi_D^* S \wedge \pi_B^* T)\right] && (\text{FR}) \\
&\cong \exists_{\pi_B}\left[\pi_D^* R \wedge \pi_A^* \exists_{\pi_C}(\pi_D^* S \wedge \pi_B^* T)\right] && (\text{B-C})
\end{aligned}
$$

Connecting Fibrations and Double Categories

Examples

EEF p	$\mathbb{B}\mathrm{il}(p)$
Codomain fibration $\mathcal{B}^{\to} \to \mathcal{B}$ ($\mathcal{B}$ a category with finite limits)	$\mathbb{S}\mathrm{pan}(\mathcal{B})$
Subobject fibration $\mathrm{Pred} \to \mathrm{Set}$ (could use $\mathrm{Sub}(\mathcal{R}) \longrightarrow \mathcal{R}$ for $\mathcal{R}$ any regular category)	$\mathbb{R}\mathrm{el}^{\circ}$

Prop. (Nasu 2025) There is a biequivalence

$$\begin{tikzcd} \mathrm{EEF} \arrow[r, bend left, "\mathbb{B}\mathrm{il}/\mathbb{F}\mathrm{r}"] \arrow[r, phantom, "\simeq"] & \mathrm{Eqp}_{\mathrm{Frob}} \arrow[l, bend left, "\mathrm{uni}"] \end{tikzcd}$$

between suitable 2-categories of EEFs and Frobenius cartesian equipments.

Extending to Coherent Fibrations

Prop. (M.) The biequivalence $\mathrm{EEF} \simeq \mathrm{Eqp}_{\mathrm{Frob}}$ restricts to a biequivalence

$$\begin{tikzcd} \mathrm{CohFib} \arrow[r, bend left, "\mathbb{B}\mathrm{il}/\mathbb{F}\mathrm{r}"] \arrow[r, phantom, "\simeq"] & \mathrm{CohEqp} \arrow[l, bend left, "\mathrm{uni}"] \end{tikzcd}$$

between suitable 2-categories of coherent fibrations and so-called coherent equipments, that is, Frobenius cartesian equipments with local distributive finite coproducts.

Conjecture The double categories termed "double Lawvere theories" by Paré in 2009 are examples of coherent equipments.

Adding Negation

First: what kind of negation are we talking about?

<u>Intuitionistic Logic</u>

- $\neg\varphi := \varphi \Rightarrow \bot$
- notably, does not satisfy excluded middle

<u>Classical Logic</u>

- $\neg\varphi$: (unique) proposition s.t.
  $\varphi \wedge \neg\varphi \equiv \bot$ and $\varphi \vee \neg\varphi \equiv \top$

- adding negation to coherent logic recovers all of classical logic, so we take this direction for now
- keep intuitionistic logic in mind for later

Dual Fibrations

Definition Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with corresponding indexed category $p^{-1} : \mathcal{B} \to \mathcal{C}at$. The dual fibration $p^{\circ} : \mathcal{E}^{\circ} \to \mathcal{B}$ of p is the fibration associated with the composite

$$\mathcal{B} \xrightarrow{p^{-1}} \mathcal{C}at \xrightarrow{(-)^{op}} \mathcal{C}at^{co}.$$

In elementary terms, $\mathcal{E}^{\circ}$ is a category with the same objects as $\mathcal{E}$, and morphisms (equivalence classes of) spans $L \xleftarrow{v} X \xrightarrow{h} R$ where v is vertical and h is cartesian.

Prop. (Kock 2015) There is a canonical isomorphism $\mathcal{E} \cong \mathcal{E}^{\circ\circ}$ over $\mathcal{B}$.

Why Dual Fibrations?

Lemma (M.) Let $p\colon \mathcal{E} \to \mathcal{B}$ be a locally preordered first-order fibration. Then there exists a morphism of fibrations $\neg : p \to p^{\bullet}$ induced by the assignment $x \mapsto \neg x$. If, moreover, p is a Boolean fibration, then there is a natural iso

$$\begin{tikzcd}
 & p^{\bullet} \arrow[dr, bend left, "\neg^{\bullet}"] \arrow[d, phantom, "\Uparrow\cong"] & \\
p \arrow[ur, bend left, "\neg"] \arrow[rr, bend right, "\cong"'] & {} & p^{\bullet\bullet}
\end{tikzcd}$$

mimicking the double negation law $\varphi \equiv \neg\neg\varphi$.

Why Dual Fibrations? (cont'd)

Proof Sketch Let $u: x \to y$ be a morphism in $\mathcal{E}$ over $f: A \to B$; u factors as $x \xrightarrow{v} f^*y \xrightarrow{h} y$ with v vertical ($p(v) = 1_A$) and h cartesian. Then $\neg u$ is given by the equivalence class of spans represented by

$$\neg x \xleftarrow{\neg v} \neg f^*y \cong f^*\neg y \xrightarrow{\neg h} \neg y$$

f^* preserves exponents and initial objects

$\neg f^*y \leq \neg x$ iff $(f^*y \Rightarrow \bot) \wedge x \leq \bot$

$(f^*y \Rightarrow \bot) \wedge x \leq (f^*y \Rightarrow \bot) \wedge f^*y \leq \bot$

Identity preservation follows from preorderedness.

Composition preservation is straightforward since morphisms in p° are equivalence classes.

Some Questions

i) Given the connection between fibrations and equipments, what is a "dual equipment"?

ii) What kinds of (virtual) double categories have duals? How is the dual constructed from the original?

iii) To what extent can this "negation morphism" be lifted to (virtual) double categories via $\mathbb{B}\mathrm{il}$?

Dual Equipments: A Motivating Example

Definition We denote by $\mathbb{R}\mathrm{el}^\circ$ the following double category:

- objects are sets;
- tight arrows are functions;
- loose arrows $R : A \nrightarrow B$ are relations $R \subseteq A \times B$;
- a cell $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "R"', "|" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "S", "|"' marking] \\ B \arrow[r, "g"'] & D \end{tikzcd}$ exists iff $(f(a), g(b)) \in S \Rightarrow (a,b) \in R$ for all $(a,b) \in A \times B$;
- given $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$, the composite $R \oplus S$ is given by

$$R \oplus S := \{ (a,c) \mid \forall b \in B.\ R(a,b) \vee S(b,c) \};$$

- the loose identity id_A on A is given by

$$\mathrm{id}_A := \{ (a,a') \in A \times A \mid a \neq a' \}.$$

Dual Equipments: A Motivating Example (cont'd)

Lemma (M.) $\mathbb{R}\mathrm{el}^{\bullet}$ is a Frobenius cartesian equipment, with companion and conjoint of a function $f: A \to B$ given by $f_* = \{(a,b) \mid b \neq f(a)\}$ and $f^* = \{(b,a) \mid b \neq f(a)\}$, respectively.

Lemma (M.) $\mathbb{R}\mathrm{el}^{\bullet} \cong \mathbb{B}\mathrm{il}(\mathrm{Pred}^{\bullet})$, where Pred denotes the subobject fibration on Set.

Lemma (M.) Taking complements of relations induces a double functor (in fact an equivalence of double categories) $\neg : \mathbb{R}\mathrm{el}^{\circ} \to \mathbb{R}\mathrm{el}^{\bullet}$. This functor is the image under $\mathbb{B}\mathrm{il}$ of the similarly-defined morphism of fibrations $\neg : \mathrm{Pred} \to \mathrm{Pred}^{\bullet}$.

Dualizing (Virtual) Double Categories

The most general version of $\mathbb{B}\mathrm{il}$ sends cartesian fibrations ($\mathcal{B}$ has finite products, $\mathcal{E}$ has fibred finite products) to <u>cartesian fibrational virtual double categories</u>.

<u>Conjecture</u> Applying $\mathbb{B}\mathrm{il}$ to a cartesian fibration with finite fibred coproducts yields a dualizable cartesian fibrational virtual double category.

Lifting Negation via $\mathbb{B}$il

Remark For $\neg : p \to p^\circ$ to be liftable to proper double categories, p° must also be an elementary existential fibration, so p must have properties and structure that makes it so.

Definition We call a fibration $p : \mathcal{E} \to \mathcal{B}$ <u>separated universal</u> if:

- $\mathcal{B}$ has finite products;
- $\mathcal{E}$ has fibred finite coproducts;
- for every parameterized diagonal $\Delta : X \times Y \to X \times Y \times Y$ in $\mathcal{B}$, the reindexing functor $\Delta^* : \mathcal{E}_{X \times Y \times Y} \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_\Delta$ satisfying dual Beck-Chevalley and Frobenius;
- for every projection $\pi_Y : X \times Y \to X$ in $\mathcal{B}$, the reindexing functor $\pi_Y^* : \mathcal{E}_X \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_\pi$ satisfying dual Beck-Chevalley and Frobenius.

Future Directions

- Equipments can be cleanly characterized using two-sided fibrations, which also have dual notions. Can these be used to model other sorts of negation?
  ↳ Paré's (co)retrocells should play a role in this story

Corollary (M.) Let $\mathbb{D}$ be a double category. Then $\mathbb{D}$ is an equipment iff the following equivalent conditions hold:

i) $\mathbb{D}_1 \xleftarrow{s} \mathbb{D}_0 \xrightarrow{t} \mathbb{D}_1$ is a jointly fibrant span;

ii) $\mathbb{D}_1 \xleftarrow{s} \mathbb{D}_0 \xrightarrow{t} \mathbb{D}_1$ is a jointly opfibrant span;

iii) $\mathbb{D}_1 \xleftarrow{s} \mathbb{D}_0 \xrightarrow{t} \mathbb{D}_1$ is a two-sided fibration and a two-sided opfibration.

- What is the internal logic of an elementary existential separated universal fibration?

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Thank you!

uni Construction

Definition (Nasu 2025) Let $\mathbb{D}$ be a cartesian equipment. Define $\mathrm{uni}(\mathbb{D})$ to be the fibration obtained from the pullback

$$\begin{tikzcd} \mathrm{Uni}(\mathbb{D}) \arrow[r] \arrow[d, "\mathrm{uni}(\mathbb{D})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, "\langle s,t \rangle"] \\ \mathbb{D}_0 \arrow[r, "A \mapsto (A{,}*)"'] & \mathbb{D}_0 \times \mathbb{D}_0 \end{tikzcd}$$

Lemma (Nasu 2025) Let $\mathbb{D}$ be a Frobenius cartesian equipment. Then $\mathbb{B}\mathrm{il}(\mathrm{uni}(\mathbb{D})) \simeq \mathbb{D}$.

Proof sketch Frobenius condition allows 'transfer of variables'; in particular, every loose arrow $R: A \nrightarrow B$ is isomorphic to a loose arrow $R': A \times B \nrightarrow *$, so that pulling back to $\mathrm{uni}(\mathbb{D})$ does not "lose information".

Theorem (Nasu 2025) There is a biequivalence between appropriate 2-categories of EEFs and Frobenius cartesian equipments.

Retrocells

Definition (Paré 2024) Let $\mathbb{D}$ be a double category with chosen companions. A retrocell in $\mathbb{D}$ (left) corresponds to a (standard) double cell of the form (right) in $\mathbb{D}$:

$$
\begin{tikzcd}
A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \Nearrow"] & C \arrow[d, "g"] \\
B \arrow[r, "S"', "|" marking] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f_*", "|" marking] \arrow[d, equal] & B \arrow[r, "S", "|" marking] \arrow[d, phantom, "\alpha"] & D \arrow[d, equal] \\
A \arrow[r, "R"', "|" marking] & C \arrow[r, "g_*"', "|" marking] & D
\end{tikzcd}
$$

Theorem (Paré) The objects, tight and loose arrows of $\mathbb{D}$, together with retrocells, form a double category $\mathbb{D}^{ret}$.

Theorem (Paré) (1) $\mathbb{D}^{ret}$ has a canonical choice of companions (namely, the companions of $\mathbb{D}$).

(2) There is a canonical isomorphism of double categories with companions

$$\mathbb{D} \xrightarrow{\cong} \mathbb{D}^{ret\ ret}$$

which is the identity on objects and tight and loose arrows.

Coretrocells

Definition (Paré 2024) Let $\mathbb{D}$ be a double category with chosen conjoints. A *coretrocell* in $\mathbb{D}$ (left) is a cell of the form (right) in $\mathbb{D}$:

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] \arrow[ur, phantom, "\Nwarrow\alpha"] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "g^*", "\shortmid"{marking}] & C \arrow[d, equal] \\ B \arrow[r, "f^*"', "\shortmid"{marking}] & A \arrow[r, "R"', "\shortmid"{marking}] \arrow[u, phantom, "\alpha"] & C \end{tikzcd}$$

Theorem (Paré) (1) The objects, tight and loose arrows of $\mathbb{D}$, together with coretrocells, form a double category $\mathbb{D}^{crt}$.

(2) $\mathbb{D}^{crt}$ has a canonical choice of conjoints.

(3) There is a canonical isomorphism of double categories with conjoints

$$\mathbb{D} \xrightarrow{\cong} \mathbb{D}^{crt\ crt}$$

which is the identity on objects and tight and loose arrows.

Considering Both Sides: Two-Sided Fibrations

Definition (Street) A span $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$ in a representable 2-category $\mathcal{K}$ is a two-sided fibration if it is a (Φ_A, Φ_B)-bimodule, where Φ_A is the comma object of the identity cospan on A, seen as a pseudomonad in $\mathrm{Span}(\mathcal{K})$.

In Cat, this means:

(i) every morphism $f: a \to px$ in B has a p-cartesian lift $f^*a \to x$ that is q-vertical;

(ii) every morphism $g: qx \to b$ in A has a q-opcartesian lift $x \to g_*b$ that is p-vertical;

(iii) for every cartesian-opcartesian composite $f^*a \longrightarrow x \longrightarrow g_*b$, the canonical morphism $g_*f^*a \longrightarrow f^*g_*b$ is an isomorphism.

A span $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$ is a two-sided opfibration if the opposite span $B \xleftarrow{p} \mathcal{E} \xrightarrow{q} A$ is a two-sided fibration.

Two-Sided Fibrations (cont'd)

Remark Given a two-sided fibration $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$, every morphism $I \to J$ in $\mathcal{E}$ factors into three:

$$I \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} J$$

where α is q-opcartesian and p-vertical, β is p,q-vertical, and γ is p-cartesian and q-vertical.

Definition (von Glehn) Let $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$ be a two-sided fibration with corresponding indexed category $(p,q)^{-1} : A^{op} \times B \to \mathrm{Cat}$. Its dual two-sided fibration $A \xleftarrow{q^\circ} \mathcal{E}^\circ \xrightarrow{p^\circ} B$ is the two-sided fibration associated with the composite $A^{op} \times B \xrightarrow{(q,p)^{-1}} \mathrm{Cat} \xrightarrow{(-)^{op}} \mathrm{Cat}^{co}$.

In elementary terms, $\mathcal{E}^\circ$ has the same objects as $\mathcal{E}$, and morphisms (equivalence classes of) diagrams

$$I \xrightarrow{\alpha} \bullet \xleftarrow{\beta^\circ} \bullet \xrightarrow{\gamma} J$$

where α is q-opcartesian and p-vertical, β° is p,q-vertical, and γ is p-cartesian and q-vertical.

Two-Sided Fibrations (cont'd)

Lemma (Capucci) A double category has companions (resp. conjoints) iff its underlying source-target span is a two-sided opfibration (resp. fibration).

Lemma (M.) Let $\mathbb{D}$ be a double category with chosen companions (resp. conjoints). The underlying source-target span of $\mathbb{D}^{\mathrm{ret}}$ (resp. $\mathbb{D}^{\mathrm{crt}}$) is (isomorphic to) the dual two-sided fibration of that of $\mathbb{D}$.

Proof Sketch Seeing cells in $\mathbb{D}$ as morphisms in the total category of a two-sided opfibration, every cell can be factored as on the right. Taking the dual two-sided fibration amounts to flipping the middle cells, which become precisely retrocells.

$$\begin{tikzcd}
A \arrow[rr, "R", "|"{marking}] \arrow[d, equal] & & C \arrow[d, "g"] \\
A \arrow[r, "R", "|"{marking}] \arrow[d, equal] \arrow[rr, phantom, "\Downarrow", shift right=4] & C \arrow[r, "g_*", "|"{marking}] & D \arrow[d, equal] \\
A \arrow[r, "f_*", "|"{marking}] \arrow[d, "f"'] \arrow[rr, phantom, "\Downarrow", shift left=4] & B \arrow[r, "S", "|"{marking}] & D \arrow[d, equal] \\
B \arrow[rr, "S"', "|"{marking}] \arrow[rr, phantom, "\Downarrow", shift left=4] & & D
\end{tikzcd}$$

Formalizing (Categorical) Duality

Def'n (Shulman 2016) A **strong duality involution** on a strict 2-category $\mathcal{A}$ consists of:

- a strict 2-functor $(-)^\bullet : \mathcal{A}^{co} \longrightarrow \mathcal{A}$;
- a strict 2-natural isomorphism η

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, equal] \arrow[dr, "((-)^\bullet)^{co}"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{A} \\
& \mathcal{A}^{co} \arrow[ur, "(-)^\bullet"'] &
\end{tikzcd}$$

- an identity modification ζ

$$\begin{tikzcd}
\mathcal{A}^{co} \arrow[r, "(-)^\bullet"] & \mathcal{A} \arrow[rr, equal] \arrow[dr, "((-)^\bullet)^{co}"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{A} \\
& & \mathcal{A}^{co} \arrow[ur, "(-)^\bullet"'] &
\end{tikzcd}
\quad \overset{\zeta}{\Rrightarrow} \quad
\begin{tikzcd}
\mathcal{A}^{co} \arrow[rr, equal] \arrow[dr, "(-)^\bullet"'] & \arrow[d, phantom, "\Downarrow\eta^{co}"] & \mathcal{A}^{co} \arrow[r, "(-)^\bullet"] & \mathcal{A} \\
& \mathcal{A} \arrow[ur, "((-)^\bullet)^{co}"'] & &
\end{tikzcd}$$

whose components are therefore identity 2-cells $\zeta_x : \eta_{x^\bullet} \overset{=}{\Rrightarrow} (\eta_x)^\bullet$.

Formalizing (Categorical) Duality (cont'd)

Examples

(i) "With nearly any reasonable set-theoretic definitions of 'category' and 'opposite'," the 2-category Cat of categories and functors has a strong (indeed strict) duality involution.

(ii) The 2-category Fib_B of fibrations over a fixed base category B has a strong duality involution sending fibrations to their duals.

A Notion of Self-Duality

Definition Let S be a sub-2-category of Fib_B. We say that S is *conceptually self-dual* if the duality involution on Fib_B restricts to a duality involution on S.

Examples Assume B has finite limits throughout.

(i) The sub-2-category of Boolean hyperdoctrines and appropriately structure-preserving morphisms is conceptually self-dual.

(ii) The analogous sub-2-category of EEFs is not conceptually self-dual, since the dual fibration of an EEF is not an EEF.

(iii) Fib_B itself is conceptually self-dual.

Since EEFs are not self-dual, $\mathbb{B}\mathrm{il}(p^\circ)$ is not definable for p an EEF.

So what are the duals to EEFs?

A Swath of Dual Definitions

Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration. We say that p is *coprimary* if:

(i) $\mathcal{B}$ has finite products;

(ii) $\mathcal{E}$ has fibred finite coproducts.

Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a coprimary fibration. We say that p is *universal* if, for every projection $\pi_Y: X \times Y \to X$ in $\mathcal{B}$, the reindexing functor $\pi_Y^*: \mathcal{E}_X \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_{\pi_Y}$ satisfying (dual) Beck-Chevalley and (dual) Frobenius reciprocity.

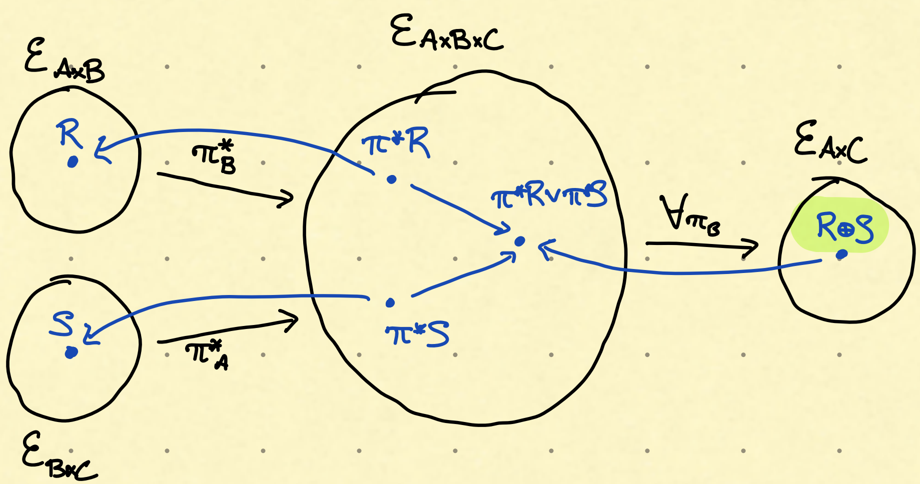
Definition Let $p: \mathcal{E} \to \mathcal{B}$ be a coprimary fibration. We say that p is *distinct* if, for every parameterized diagonal $\Delta: X \times Y \to X \times Y \times Y$ in $\mathcal{B}$, the reindexing functor $\Delta^*: \mathcal{E}_{X \times Y \times Y} \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_\Delta$ satisfying (dual) Beck-Chevalley and (dual) Frobenius reciprocity.

Upshot p distinct universal fibration $\iff$ p° elementary existential fibration

A Dual $\mathbb{B}$il Construction

<u>Proposition (M.)</u> Let $p:\mathcal{E}\to\mathbb{B}$ be a distinct universal fibration. The following data form a double category $\mathbb{B}\mathrm{il}^{\circ}(p)$:

- the tight part of $\mathbb{B}\mathrm{il}^{\circ}(p)$ is $\mathbb{B}$;
- loose arrows $A \nrightarrow B$ are objects in the fibre $\mathcal{E}_{A\times B}$;
- cells $\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$ in $\mathbb{B}\mathrm{il}^{\circ}(p)$ are given by morphisms $\alpha: R\to S$ in $\mathcal{E}$ over $f\times g$;
- loose identity on A is given by $\forall_{\Delta_A}(\perp_A)\in\mathcal{E}_{A\times A}$, where $\perp_A$ is the initial object in $\mathcal{E}_A$;
- given loose arrows $R: A\nrightarrow B$, $S: B\nrightarrow C$, the composite $R\oplus S$ is given as below:



A Dual $\mathbb{B}il$ Construction (cont'd)

Proof Segment (Unitors) Let $R : A \nrightarrow B$ be a loose arrow in $\mathbb{B}il^\circ(p)$.

$$
\begin{aligned}
\forall_{\pi_A}\left(\pi_B^* \forall_{\Delta_A}(\bot_A) \vee \pi_A^* R\right) &\cong \forall_{\pi_A}\left(\forall_\Delta \pi_B^*(\bot_A) \vee \pi_A^* R\right) && (B\text{-}C) \\
&\cong \forall_{\pi_A}\left(\forall_\Delta(\bot_{A\times B}) \vee \pi_A^* R\right) && (\pi^* \text{ pres. } \bot) \\
&\cong \forall_{\pi_A}\left(\forall_\Delta\left(\bot_{A\times B} \vee \Delta^* \pi_A^* R\right)\right) && (FR) \\
&\cong \forall_{\pi_A}\left(\forall_\Delta\left(\Delta^* \pi_A^* R\right)\right) \\
&\cong \forall_{1_{A\times B}}\left(1_{A\times B}^* R\right) \\
&\cong R
\end{aligned}
$$

A Dual $\mathbb{B}$il Construction (cont'd)

Lemma (M.) Let p be a distinct universal fibration. Then

$$\mathbb{B}il^{\bullet}(p) \simeq \mathbb{B}il(p^{\bullet})$$

as double categories.

Conjecture (M.) Let $Fib_{EE,DU}$ denote the sub-2-category of Fib_B (for fixed B with finite products) consisting of those fibrations which are both elementary existential and distinct universal (with suitable morphisms). Then $Fib_{EE,DU}$ is conceptually self-dual.

Conjecture (M.) For any sub-2-category S of Fib_B, the (2-)pullback

$$\begin{tikzcd} \text{"}S \cap S^{\bullet}\text{"} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & S^{\bullet} \arrow[d, hook] \\ S \arrow[r, hook] & Fib_B \end{tikzcd}$$

is conceptually self-dual.

Future Work

- Characterize double categories (essentially) of the form $\mathbb{B}\mathrm{il}(p)$ for p elementary existential + distinct universal

- What is the internal logic of such fibrations?

- Observe that a cell α in an equipment can be equivalently written in 4 ways:

$$
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\alpha"] & C \arrow[r, "g_*", "\shortmid"{marking}] & D \arrow[d, equal] \\
A \arrow[r, "f_*"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrd, phantom, "\alpha'"] & A \arrow[r, "R", "\shortmid"{marking}] & C \arrow[d, equal] \\
B \arrow[r, "S"', "\shortmid"{marking}] & D \arrow[r, "g^*"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, "R", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "\alpha^r"] & & & C \arrow[d, equal] \\
A \arrow[r, "f_*"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & D \arrow[r, "g^*"', "\shortmid"{marking}] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "\alpha^l"] & A \arrow[r, "R", "\shortmid"{marking}] & C \arrow[r, "g_*", "\shortmid"{marking}] & D \arrow[d, equal] \\
B \arrow[rrr, "S"', "\shortmid"{marking}] & & & D
\end{tikzcd}
$$

and that cells in $\mathbb{R}\mathrm{el}^{\bullet}$ are precisely cells in $\mathbb{R}\mathrm{el}^{\circ}$ of the form $(\alpha^r)^{op}$. What notions of duality can we get by taking duals of the other three kinds of cells?