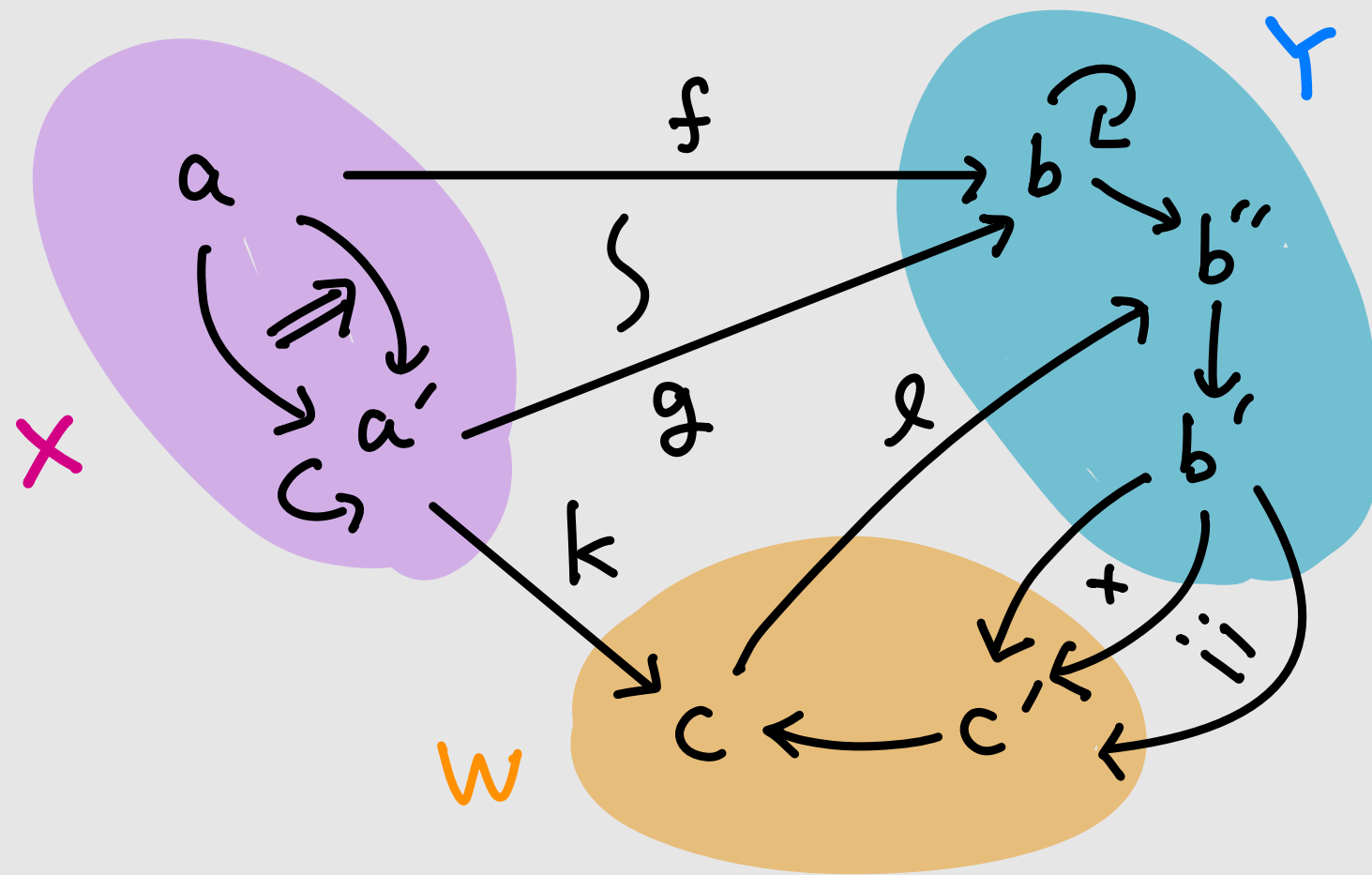


Categories enriched in a virtual double category

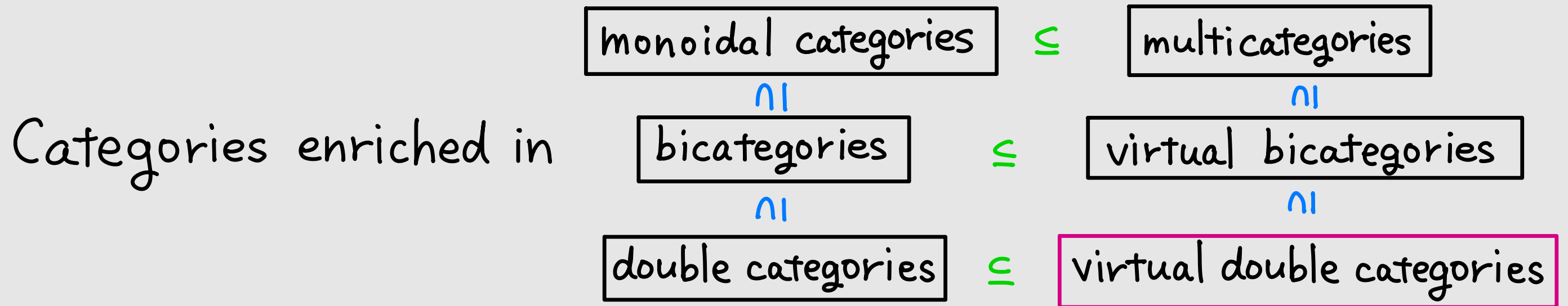


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This project started as a research project in the 2026 Adjoint School, titled
"Double Category Theoretic Perspective on Partiality."

Introduction



The only non-trivial example of double-category enrichment :

Thm (Cockett, Garner "Restriction categories as enriched categories" (2014))

restriction categories $=$ $\mathbb{P}(\text{StabLat}^{\text{op}})$ -categories

1. Enrichment in monoidal categories
2. Enrichment in bicategories
3. Enrichment in double categories
4. Lax slice enrichment
 - enrichment for restriction categories

Categories

A (locally small) category $\mathcal{C}$ consists of :

- a collection $|\mathcal{C}|$ of objects ;

- a family $(\mathcal{C}(c, d))_{c, d \in |\mathcal{C}|}$ of hom-sets ;

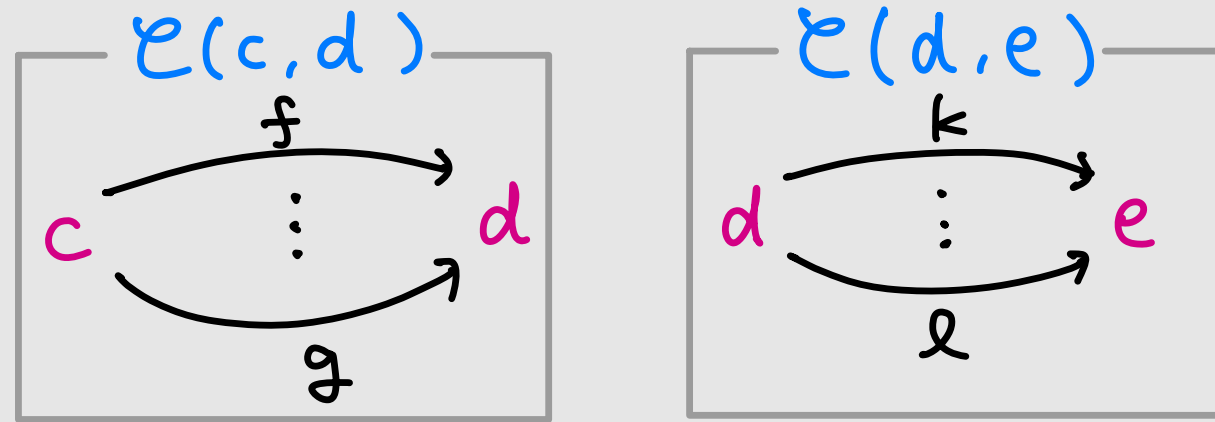
- a family of functions $(\mathcal{C}(c, d) \times \mathcal{C}(d, e) \xrightarrow{g_{c, d, e}} \mathcal{C}(c, e))_{c, d, e}$
 $(c \xrightarrow{f} d, d \xrightarrow{g} e) \mapsto c \xrightarrow[g \circ f]{f \circ g} e$
(composition)

- a family of constants $(id_c \in \mathcal{C}(c, c))_c$
(identity)

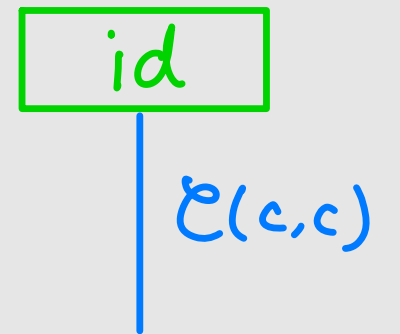
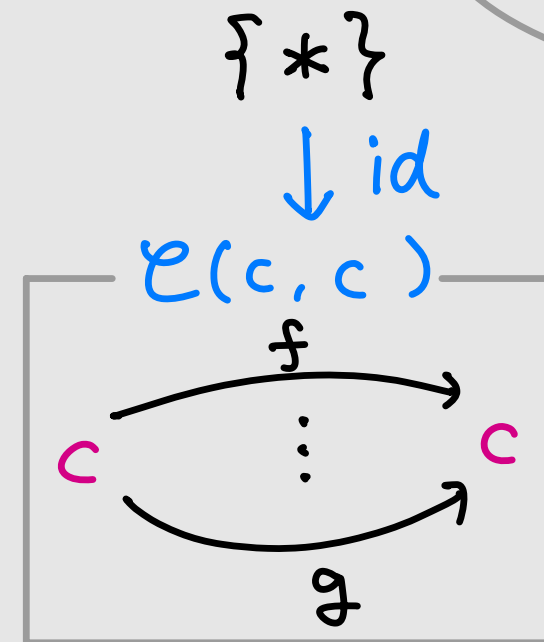
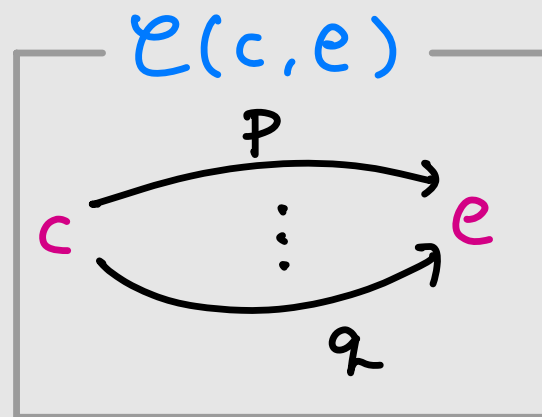
with some axioms.

String diagrams for n-ary functions

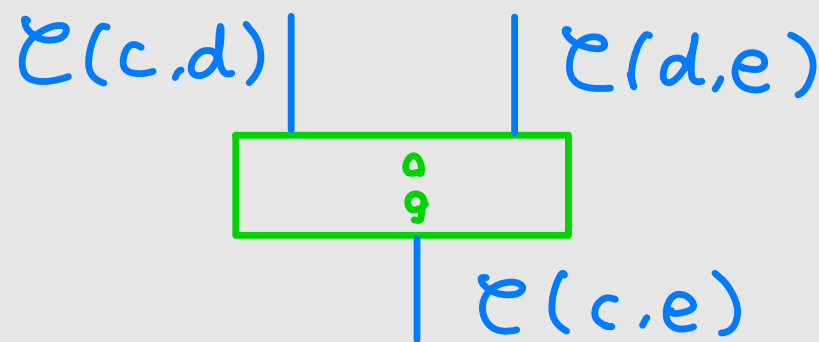
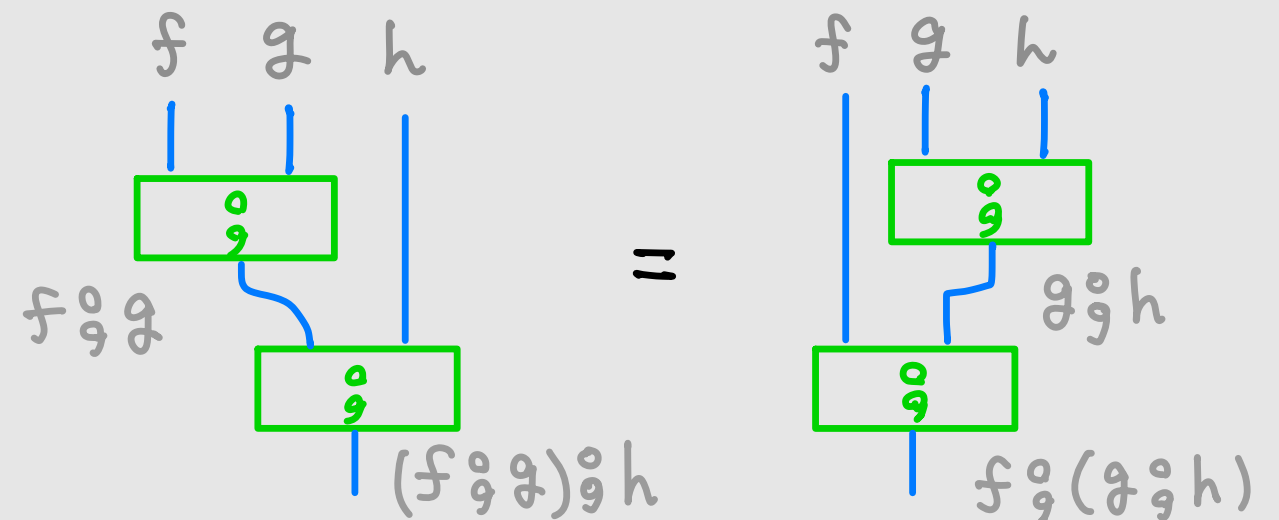
Constants are
0-ary functions!



composition $\downarrow$



Associativity $(f \circ g) \circ h = f \circ (g \circ h)$



binary
function

Categories

A (locally small) category $\mathcal{C}$ consists of :

- a collection $|\mathcal{C}|$ of objects ;

- a family $(|\mathcal{C}(c,d)|)_{c,d}$ of hom-sets ;

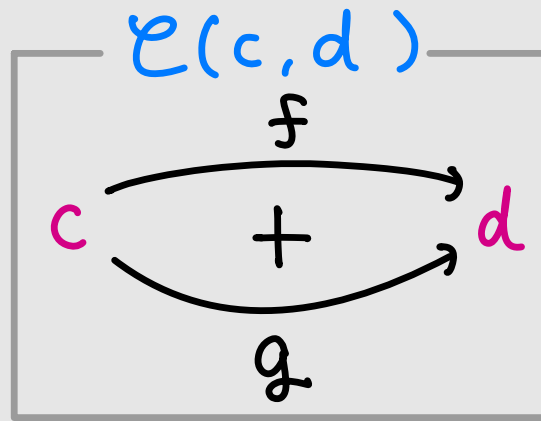
- a family of functions $\left(\begin{array}{c} \mathcal{C}(c,d) \quad | \quad \mathcal{C}(d,e) \\ \boxed{\circ} \\ \mathcal{C}(c,e) \end{array} \right)_{c,d,e}$ (composition) ;

- a family of constants $\left(\begin{array}{c} \boxed{id} \\ \mathcal{C}(c,c) \end{array} \right)_c$ (identity) ;

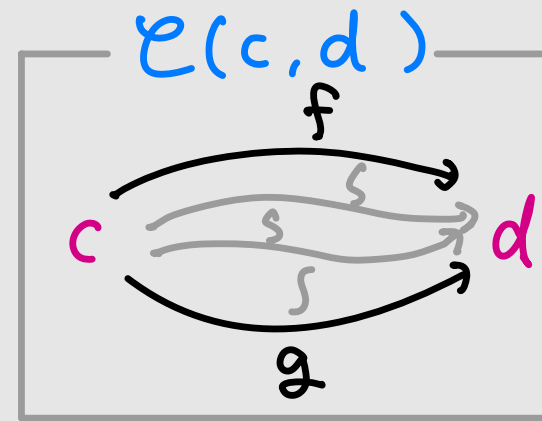
with some axioms.

Motivation of enrichment

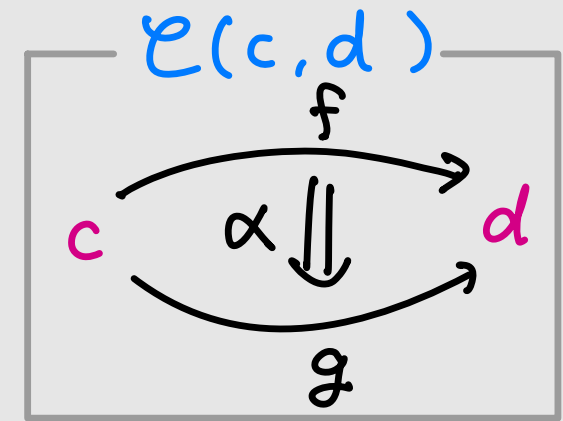
Hom-sets $\mathcal{C}(c, d)$ are sometimes more (or less) than just sets.



Hom-monoids



Hom-spaces



Hom-categories

What structure do we need for mathematical objects to form
in order to define categories enriched therein?

Multicategories

A multicategory $\mathcal{V}$ consists of: $\equiv$ virtual monoidal categories

- a collection $|\mathcal{V}|$ of objects;

$A \mid , B \mid , C \mid$

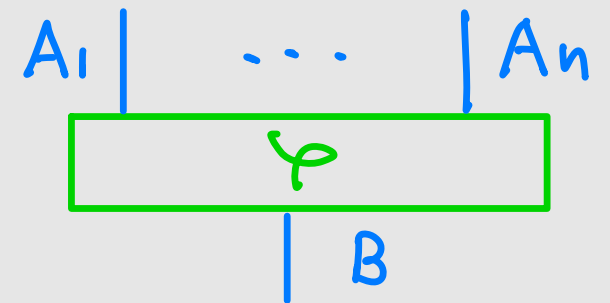
- a collection of n -ary morphisms

depending on pairs of an n -tuple of objects

$\vdots$

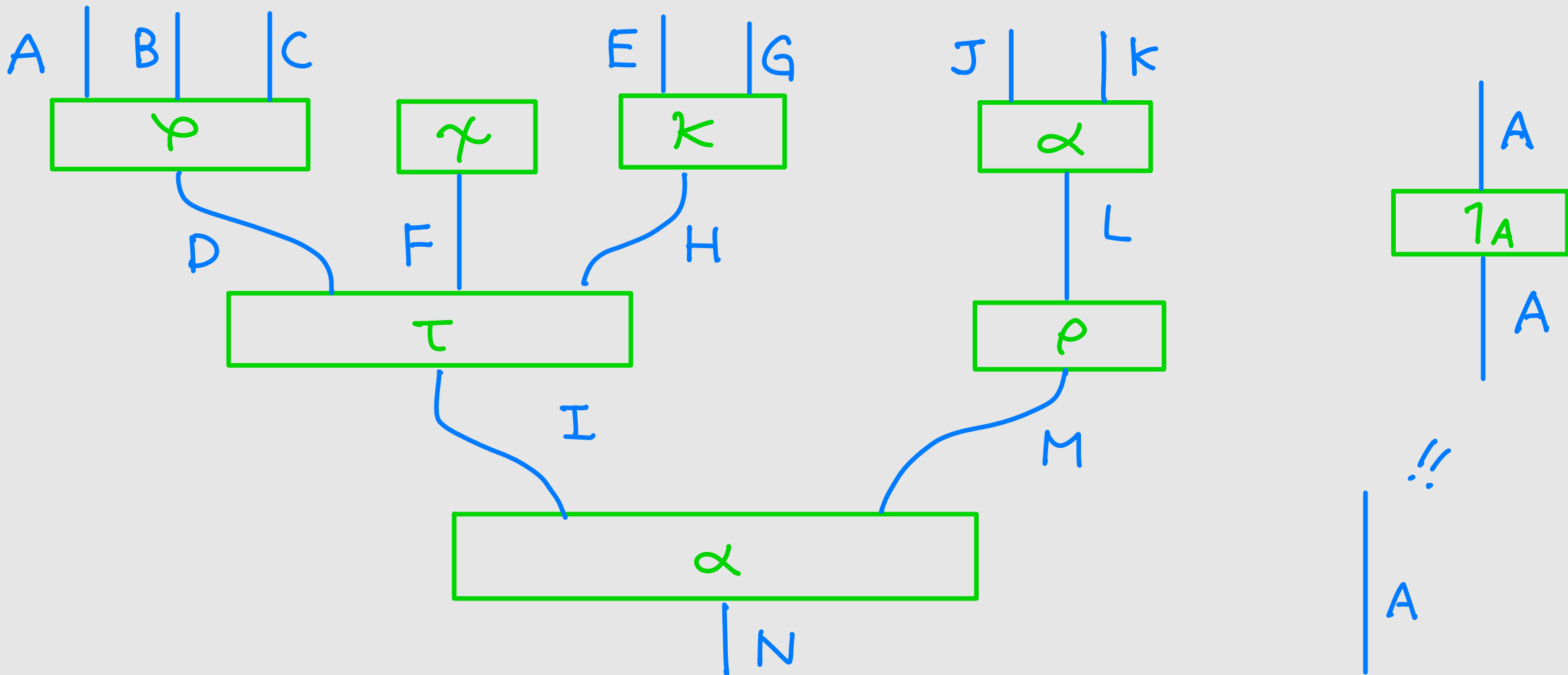
and an object

(composition, identities, some axioms)



The idea: You can compose multimorphisms in a unique way.

Composition in a multicategory



Enrichment in a multicategory

$\mathcal{V}$: a multicategory

A $\mathcal{V}$ -enriched category $\mathcal{C}$ consists of :

- a collection $|\mathcal{C}|$ of objects ;
- a family $(|\mathcal{C}(c,d)|)_{c,d \in |\mathcal{C}|}$ of **hom-objects** in $\mathcal{V}$;
- a family of binary morphisms in $\mathcal{V}$ of **composition**

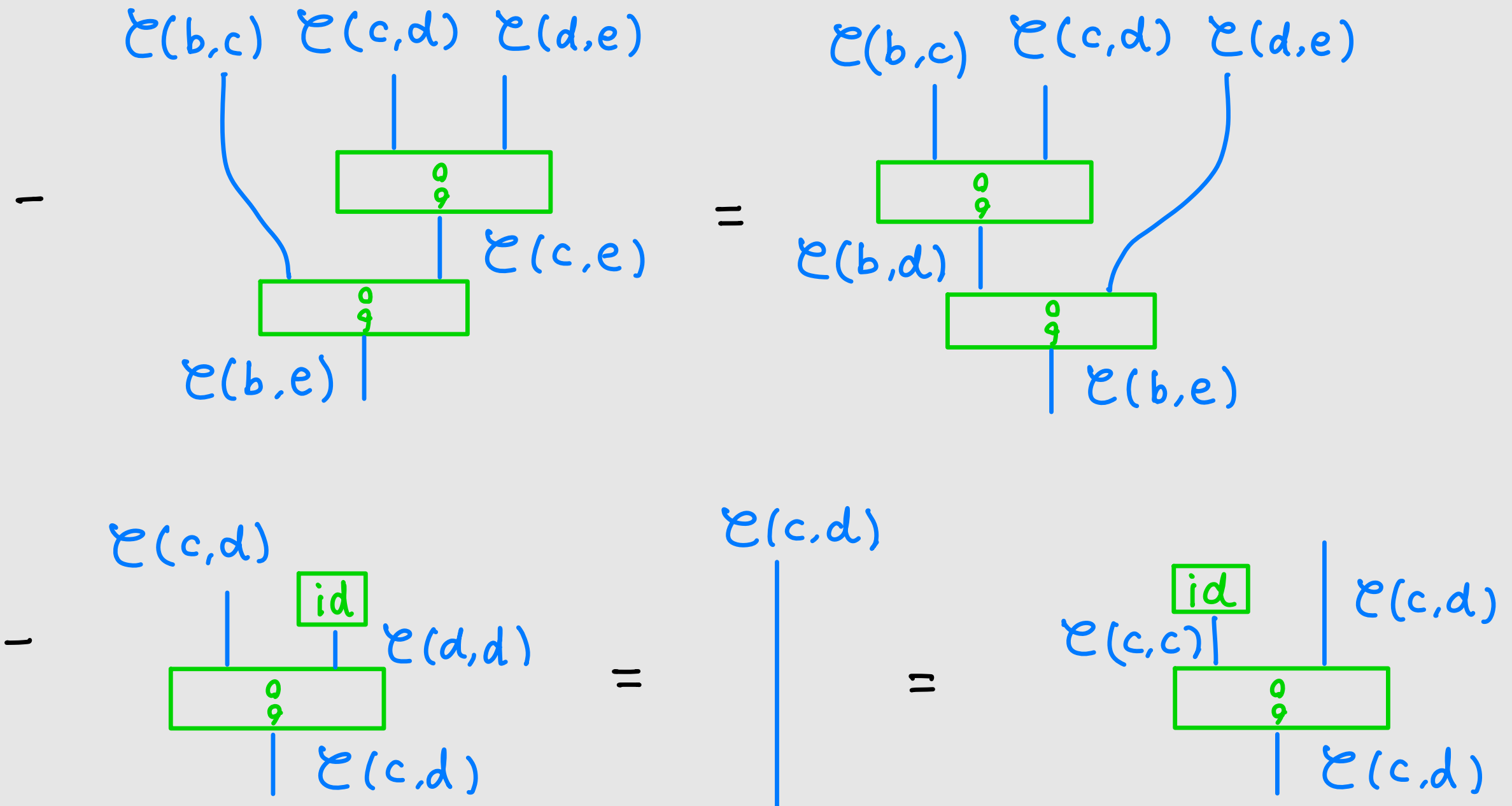
$$\left(\begin{array}{ccc} \mathcal{C}(c,d) & & \mathcal{C}(d,e) \\ & \boxed{\circ} & \\ & | & \\ & \mathcal{C}(c,e) & \end{array} \right)_{c,d,e \in |\mathcal{C}|} ;$$

- a family of nullary morphisms in $\mathcal{V}$ of **identities** ;

$$\left(\begin{array}{c} \boxed{id} \\ | \\ \mathcal{C}(c,c) \end{array} \right)_{c \in |\mathcal{C}|}$$

(ctn'd)

that satisfy the following equations in $\mathcal{L}$:



Examples

- Set :

objects : sets

n-ary morphisms : n-ary functions

$\leadsto$ Set-categories are categories

- Cat :

objects : categories

n-ary morphisms : n-ary functors

$\leadsto$ Cat-cats are 2-categories

- Ab (or Vect_k , or even Mon)

objects : Abelian groups

multi-morphisms : multi-linear maps

$\leadsto$ Ab-categories are additive cats.

- $\mathbb{2} = \{\perp \leq \top\}$

$$\exists \begin{array}{c} |t_1 \quad \dots \quad |t_n \\ \hline |s \end{array} \parallel \begin{array}{c} t_1 \wedge \dots \wedge t_n \\ \Downarrow \\ s \end{array}$$

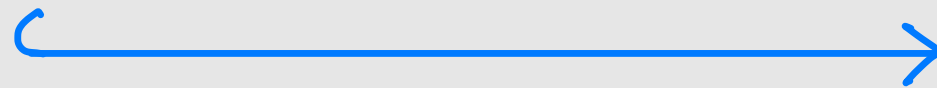
$\leadsto$ $\mathbb{2}$ -categories are preordered sets.



I know about enrichment in a **monoidal category**.

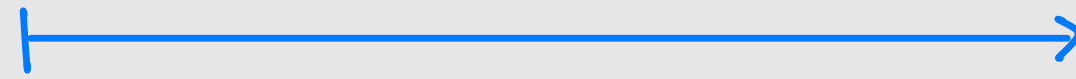
Is it different? Why don't we use that?

Monoidal categories



Multicategories

$(A, \otimes, I)$



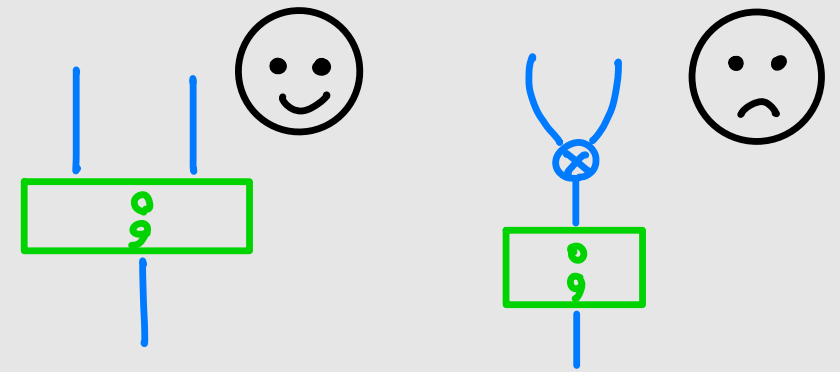
$\hat{A}$

$$\left\{ \begin{array}{c} A_1 \mid \dots \mid A_n \\ \boxed{\gamma} \\ \mid B \end{array} \text{ in } \tilde{A} \right\} := A(I - (A_1 \otimes A_2) \cdots \otimes A_n, B)$$

$(A, \otimes, I)$ -categories $\equiv \hat{A}$ -categories

Advantages of multicategories in the context of enrichment

- ① Morphisms from monoidal products are all you need to define enriched categories. **Multimorphisms** form a minimal structure for this.
- (Multicategories = Virtual monoidal categories) | | 😊 | ☹️



- ② Multicategories are a coherence-free framework!

Accordingly, the definition gets simpler.

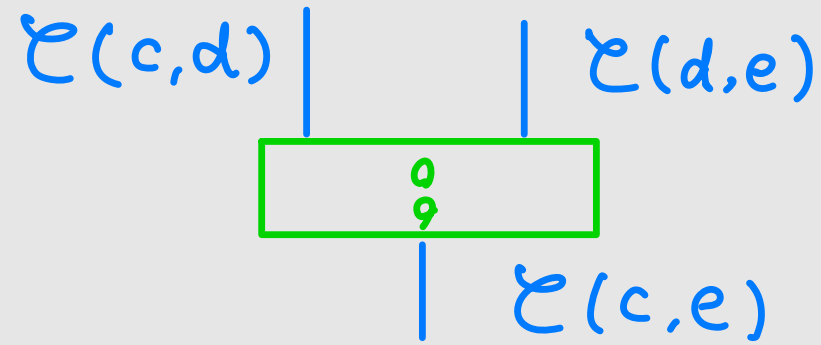
$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\cong} & (A \otimes B) \otimes (C \otimes D) \\ \text{sl} \downarrow & & \searrow \cong \\ (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\cong} & A \otimes ((B \otimes C) \otimes D) \end{array}$$

- ③ Multicategories are **more general** than monoidal categories.

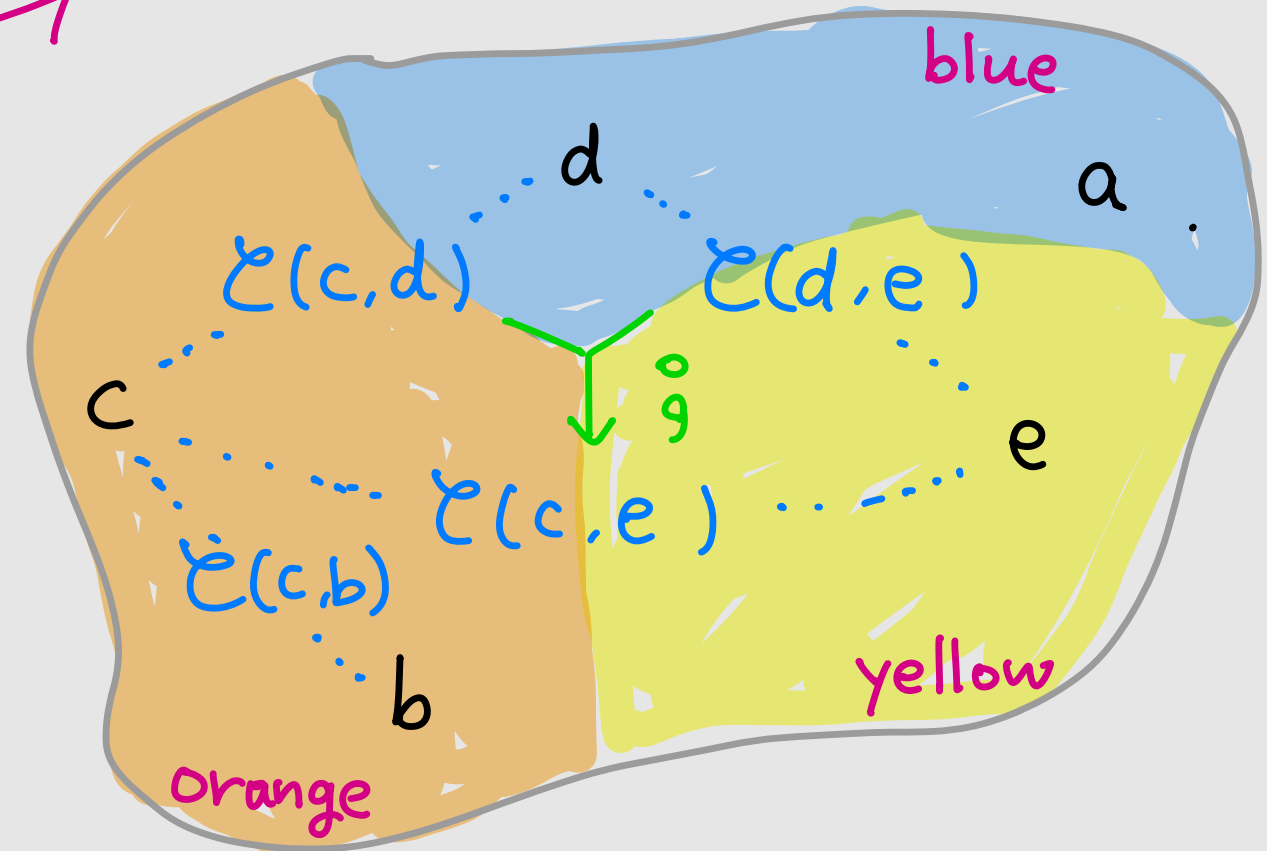
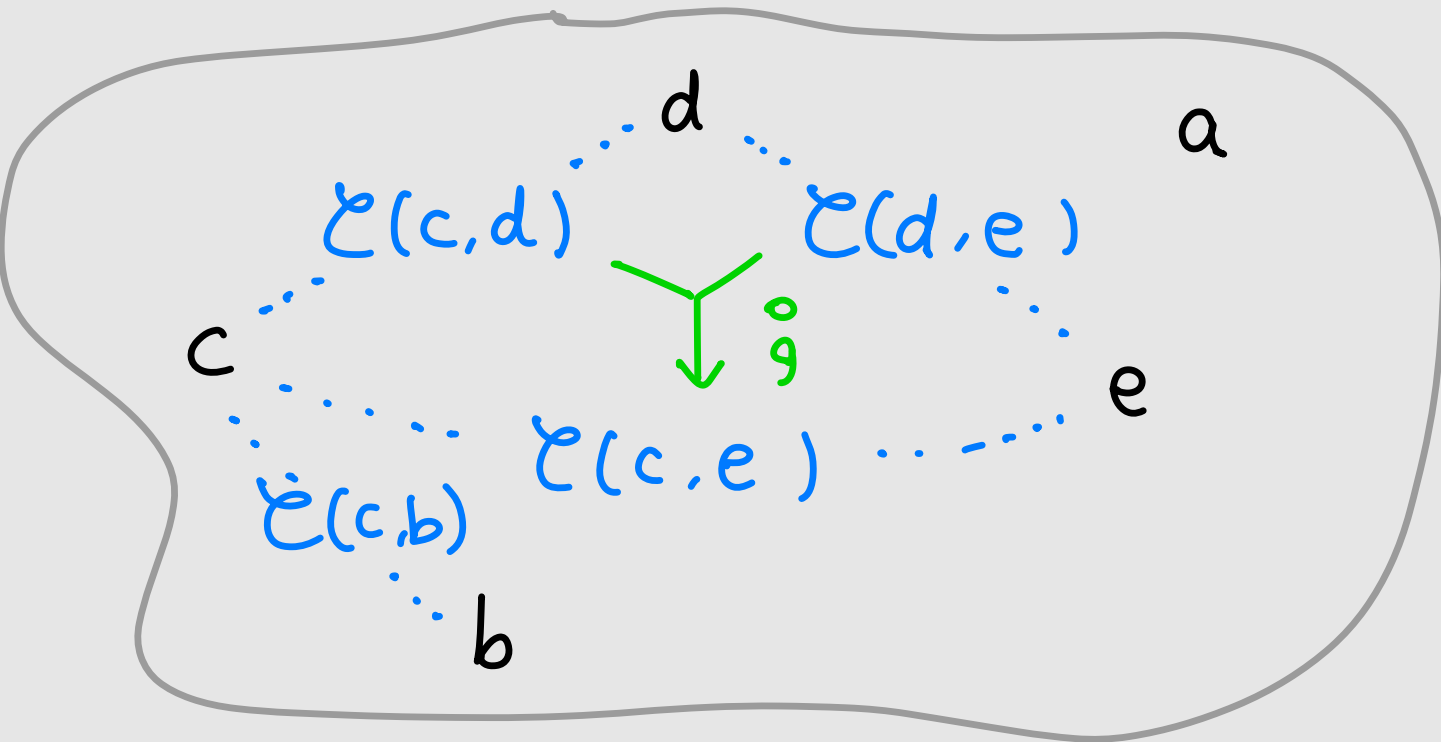
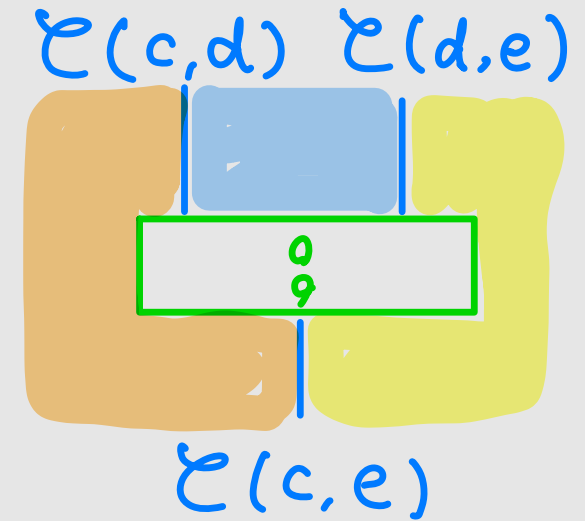
Eg. Grp is a multicategory but not a monoidal category (representable).

1. Enrichment in virtual monoidal categories
2. Enrichment in virtual bicategories
3. Enrichment in virtual double categories
4. Lax slice enrichment
 - enrichment for restriction categories

Objects with extent



generalize



Virtual bicategory

A virtual bicategory $\mathcal{V}$ consists of :

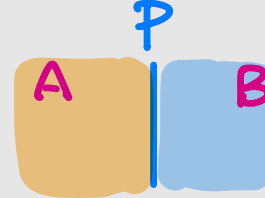
- a collection $ob\mathcal{V}$ of objects;

$A, B, C, \dots$



- a collection of 1-cells;

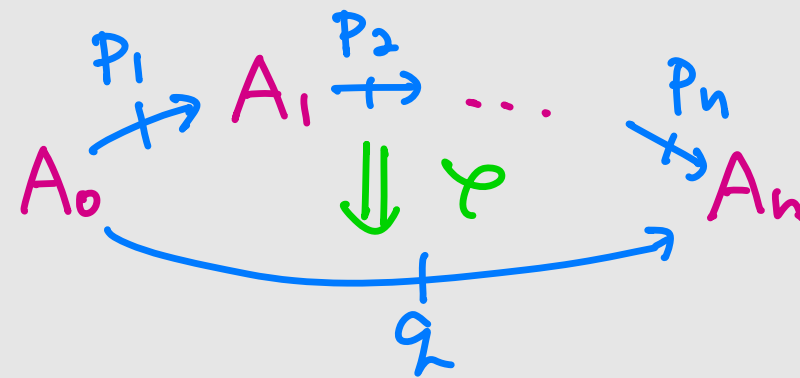
$A \xrightarrow{p} B, \dots$



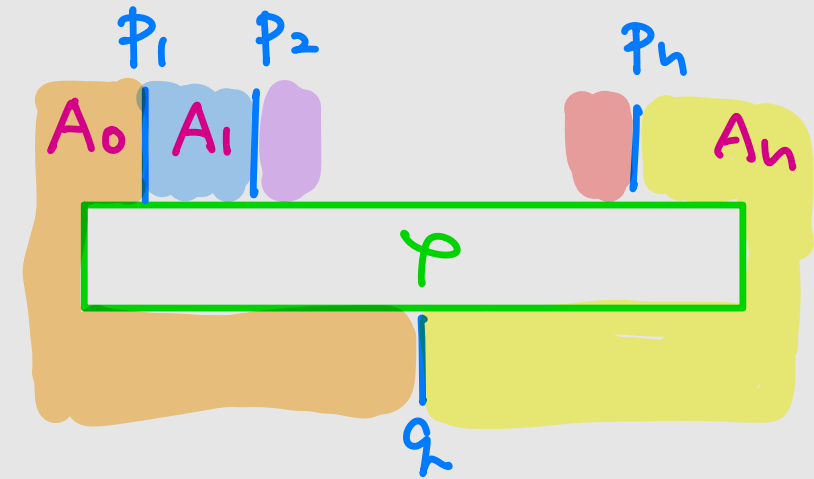
- a collection of multiary 2-cells;

⋮

$A_0 \xrightarrow{p_1} A_1 \xrightarrow{p_2} \dots \xrightarrow{p_n} A_n$



$A_0 \xrightarrow{p_1} A_1 \xrightarrow{p_2} \dots \xrightarrow{p_n} A_n$



⚠ No composition of 1-cells.

Composition of 2-cells is defined similarly to multicategories.

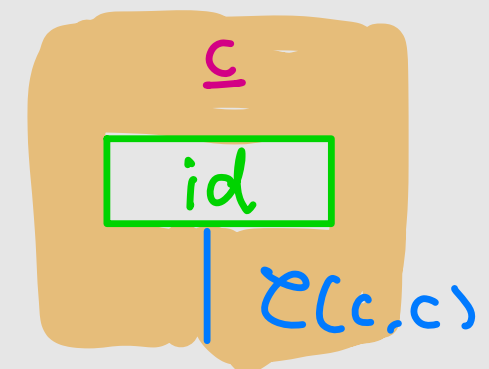
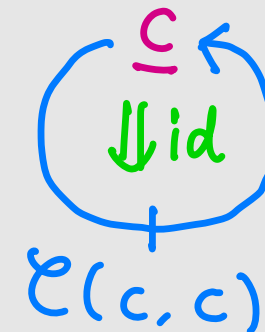
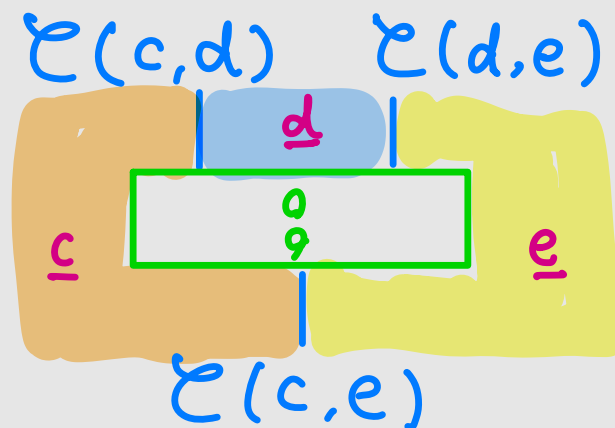
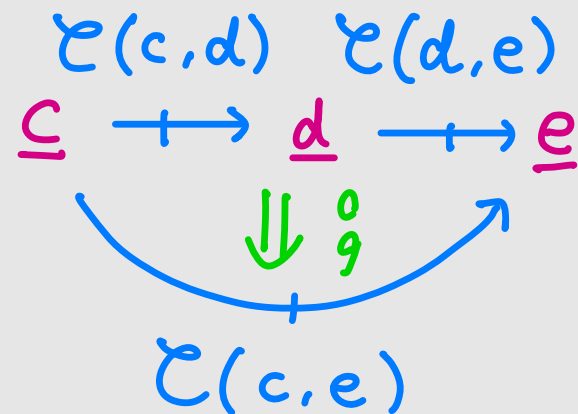
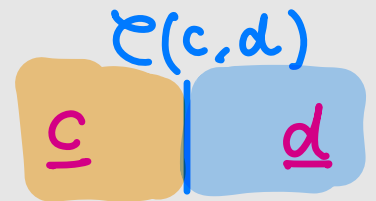
Enrichment in virtual bicategory

$\mathcal{V}$: virtual bicategory

cf. Street "Enriched categories and cohomology" (1983)

A $\mathcal{V}$ -enriched category $\mathcal{C}$ consists of :

- a collection $|\mathcal{C}|$ of objects ;
- a function $\underline{(-)} : |\mathcal{C}| \rightarrow \text{ob } \mathcal{V}$ assigning the **extent** to each object ;
- a family $(\underline{c} \xrightarrow{\mathcal{C}(c,d)} \underline{d})_{c,d \in |\mathcal{C}|}$ of **hom-1-cells** ;
- a family of 2-cells of **composition** ;
- a family of 2-cells of **identities** ;



+ some axioms

Example 1

A Rel-category $\mathcal{C}$ has

- a collection $|\mathcal{C}|$;
- a family of sets $(X_c)_{c \in |\mathcal{C}|}$;
 the extent of c $\swarrow$ $(-) = X_-$
- a family of relations $(X_c \xrightarrow{\mathcal{C}(c,d)} X_d)_{c,d \in |\mathcal{C}|}$;
 $\swarrow$

Let us write

$$x \leq y$$

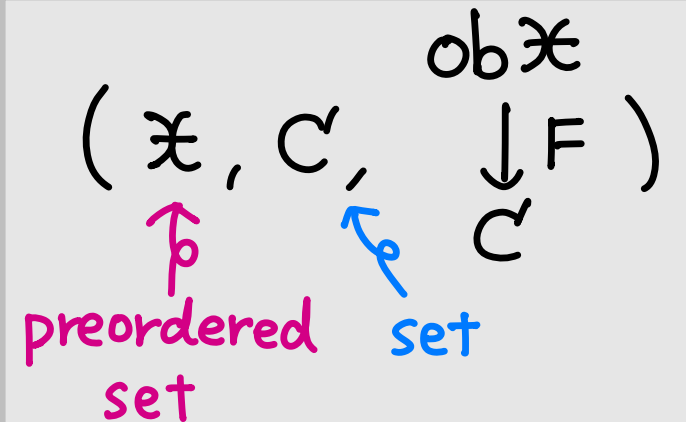
when $(x, y) \in \mathcal{C}(c, d)$

s.t.

- $x \leq y \leq z \Rightarrow x \leq z$ ($x \in X_c, y \in X_d, z \in X_e$);
- $x \leq x$ ($x \in X_c$).

This is a preorder on $\coprod_c X_c$.

$\leadsto$ A Rel-category is a partitioned preordered set.

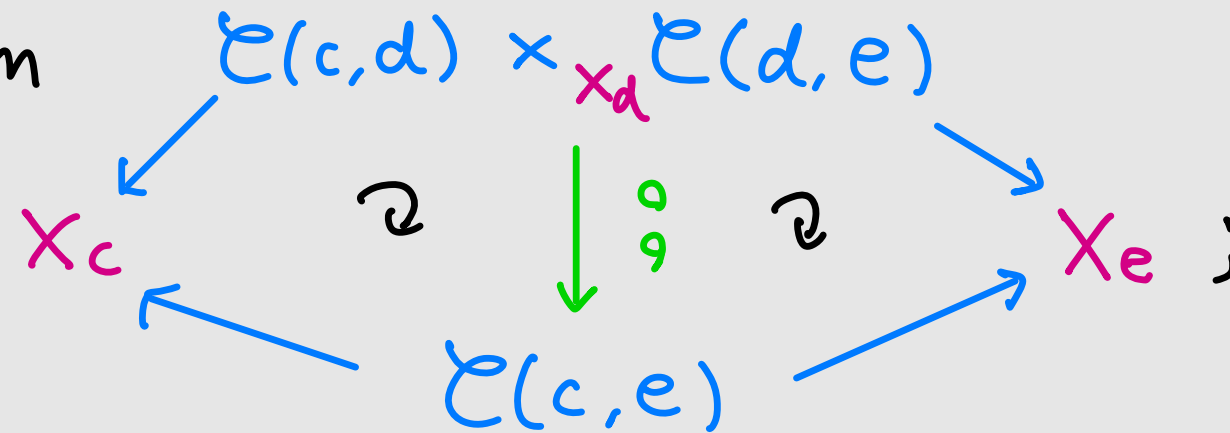


Example 2

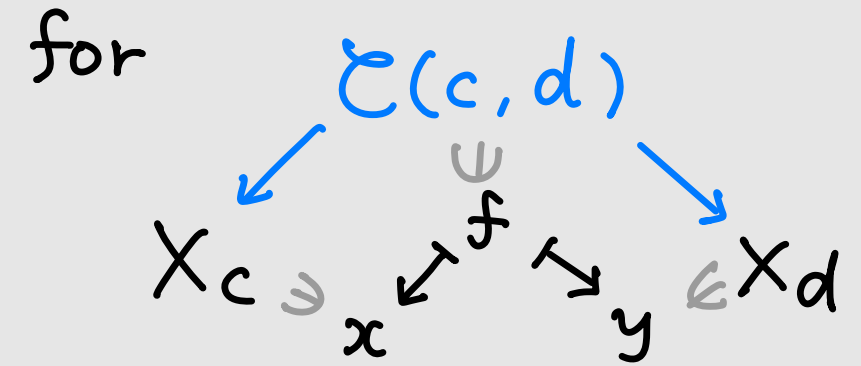
A Span-category $\mathcal{C}$ has

- a collection $|\mathcal{C}|$;
- a family of sets $(X_c)_{c \in |\mathcal{C}|}$;
- a family of spans $(X_c \xrightarrow{\mathcal{C}(c,d)} X_d)$;
- a span morphism

: (identities
+
axioms)



Let us write $x \xrightarrow{f} y$

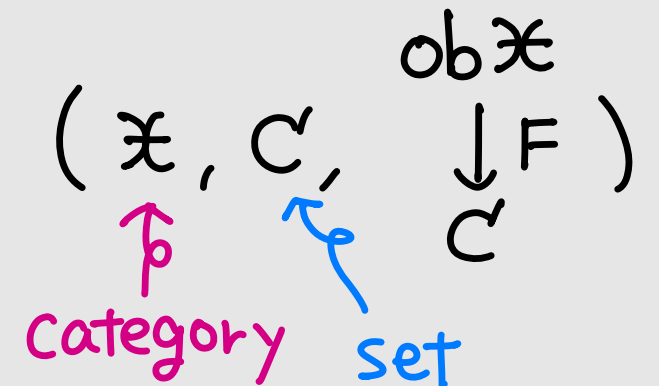


$$x \xrightarrow{f} y = y' \xrightarrow{g} z$$

$$\downarrow$$

$$x \xrightarrow{f \circ g} z$$

$\leadsto$ A Span-category is a partitioned category.



Example 2'

For a category $\mathcal{E}$ with finite limits,

a $\text{Span}(\mathcal{E})$ -category has a family of objects $(X_c)_{c \in |\mathcal{E}|}$

and compositional structure

$$\mathcal{C}(c, d) \times_{X_d} \mathcal{C}(d, e) \xrightarrow{\circ} \mathcal{C}(c, e) \quad \text{internal to } \mathcal{E}.$$

- A single-object $\text{Span}(\mathcal{E})$ -category = an $\mathcal{E}$ -internal category
- A $\text{Span}(\mathcal{E})$ -category with $X_c = 1$ ($\forall c$) = an $\mathcal{E}$ -enriched category

cf. Betti, Walters "On cocompleteness of locally-internal categories" (1987)

Example 3

L : a preordered set $\rightsquigarrow \mathcal{Q}_L$: a bicategory

objects : $a \in L$, $\mathcal{Q}_L(a, b) = \begin{cases} \{\perp \leq \top\} & \text{if } a \leq b, \\ \{\perp\} & \text{otherwise.} \end{cases}$

Composition of 1-cells is given by $\wedge$.

A $\mathcal{Q}_L$ -category $\mathcal{C}$ consists of

- a collection of objects $|\mathcal{C}|$ with the extent $|\mathcal{C}| \xrightarrow{(-)} L$;
- a binary relation $\leq$ on objects s.t. $x \leq y \Rightarrow \underline{x} \leq \underline{y}$;

that satisfies reflexivity and transitivity.

$\rightsquigarrow$ A $\mathcal{Q}_L$ -category is a preordered set C w/ an order-preserving map $C \xrightarrow{(-)} L$.

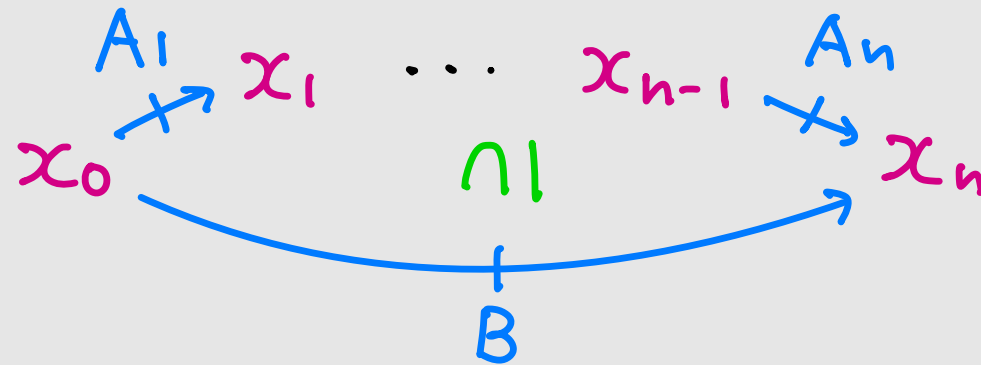
Example 4

Fix a category S . We define a (virtual) bicategory Q_S as follows.

objects : objects in S

1-cells $A : x \rightarrow y : A \subseteq S(x, y)$

2-cells



There is at most one cell.

It exists when

$$x_{i-1} \xrightarrow{f_i} x_i \in A_i \quad (\forall i)$$

$$x_0 \xrightarrow{f_1 \circ \dots \circ f_n} x_n \in B$$

A Q_S -category is a concrete category in S .

Rosenthal "Free quantaloids" (1991)

Niefield "Change of base for relational variable sets" (2004)

$$\left(\begin{array}{c} \mathcal{X} \\ \uparrow \text{category} \end{array}, \begin{array}{c} \mathcal{X} \\ \downarrow F \\ S \end{array} \right)$$

faithful

More examples

- Sheaves on a site $(\mathcal{C}, T)$ = Symmetric Cauchy-complete $\mathbf{Rel}(\mathcal{C}, T)$ -cats.
Walters "Sheaves on sites as Cauchy-complete categories" (1982)
- Fiberwise Cauchy-complete fibrations over $\mathcal{C}$ = Cauchy-complete $\mathcal{W}(\mathcal{C})$ -cats.
Betti, Carboni, Street, Walters "Variation through enrichment" (1983)
- Categories locally internal to $\mathcal{C}$ = $\mathbf{Span}(\mathcal{C})$ -categories with certain limits
Betti, Walters "On cocompleteness of locally-internal categories" (1987)
- Topological categories = Total $\mathbf{Q}_{\mathbf{Set}}$ -categories
Garner "Topological functors as total categories." (2014)

⚠ Bicategories enriched in a monoidal bicategory are a different concept.
(A connection is given in Garner, Shulman "Enriched categories as a free cocompletion". (2015))

Functors enriched in a multicategory

A $\mathcal{V}$ -functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $|F|: |\mathcal{C}| \rightarrow |\mathcal{D}|$; $\leftarrow$ object assignment

- a family $\left(\begin{array}{c} \mathcal{C}(c, c') \\ \boxed{F_{c, c'}} \\ \mathcal{D}(F_c, F_{c'}) \end{array} \right)_{c, c' \in |\mathcal{C}|}$; $\leftarrow$ morphism assignment

s.t.

$$\begin{array}{c} \mathcal{C}(c, c') \\ \downarrow F_{c, c'} \\ \mathcal{D}(F_c, F_{c'}) \end{array} \text{ in } \mathcal{V}$$

$$\begin{array}{c} \mathcal{C}(c, c') \quad \mathcal{C}(c', c'') \\ \downarrow \quad \downarrow \\ \boxed{\circ_{\mathcal{C}}} \\ \downarrow \mathcal{C}(c, c') \\ \boxed{F_{c, c'}} \\ \downarrow \\ \mathcal{D}(F_c, F_{c'')}\end{array} = \begin{array}{c} \mathcal{C}(c, c') \quad \mathcal{C}(c', c'') \\ \downarrow \quad \downarrow \\ \boxed{F_{c, c'}} \quad \boxed{F_{c', c''}} \\ \downarrow \mathcal{D}(F_c, F_{c'}) \quad \downarrow \mathcal{D}(F_{c'}, F_{c''}) \\ \boxed{\circ_{\mathcal{D}}} \\ \downarrow \\ \mathcal{D}(F_c, F_{c'')}\end{array}, \quad \begin{array}{c} \boxed{id_c} \\ \downarrow \mathcal{C}(c, c) \\ \boxed{F_{c, c}} \\ \downarrow \\ \mathcal{D}(F_c, F_c)\end{array} = \begin{array}{c} \boxed{id_{F_c}} \\ \downarrow \\ \mathcal{D}(F_c, F_c)\end{array}$$

Functors enriched in a virtual bicategory

A $\mathcal{V}$ -functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $|F|: |\mathcal{C}| \rightarrow |\mathcal{D}|$ that preserves the extent;

$$\begin{array}{ccc} |\mathcal{C}| & \xrightarrow{|F|} & |\mathcal{D}| \\ \text{(-)} \searrow & = & \swarrow \text{(-)} \\ & \text{ob } \mathcal{V} & \end{array}$$

- a family $\left(\begin{array}{c} \underline{c} \\ \parallel \\ \underline{F_c} \end{array} \begin{array}{c} \mathcal{C}(c, c') \\ \boxed{F_{c, c'}} \\ \mathcal{D}(F_c, F_{c'}) \end{array} \begin{array}{c} \underline{c'} \\ \parallel \\ \underline{F_{c'}} \end{array} \right)_{c, c' \in |\mathcal{C}|}$;

$$\begin{array}{ccc} & \mathcal{C}(c, c') & \\ & \downarrow & \\ \underline{c} & \Downarrow F_{c, c'} & \underline{c'} \\ & \uparrow & \\ & \mathcal{D}(F_c, F_{c'}) & \end{array}$$

s.t.

$$\begin{array}{c} \mathcal{C}(c, c') \quad \mathcal{C}(c', c'') \\ \downarrow \quad \downarrow \\ \boxed{\begin{array}{c} \underline{c'} \\ \circ \\ \underline{c} \end{array}} \\ \downarrow \mathcal{C}(c, c') \\ \boxed{F_{c, c'}} \\ \downarrow \\ \mathcal{D}(F_c, F_{c''}) \end{array} = \begin{array}{c} \mathcal{C}(c, c') \quad \mathcal{C}(c', c'') \\ \downarrow \quad \downarrow \\ \boxed{F_{c, c'}} \quad \boxed{F_{c', c''}} \\ \downarrow \mathcal{D}(F_c, F_{c'}) \quad \downarrow \mathcal{D}(F_{c'}, F_{c''}) \\ \boxed{\begin{array}{c} \underline{c'} \\ \circ \\ \underline{c} \end{array}} \\ \downarrow \\ \mathcal{D}(F_c, F_{c''}) \end{array}, \quad \begin{array}{c} \boxed{id_c} \\ \downarrow \mathcal{C}(c, c) \\ \boxed{F_{c, c}} \\ \downarrow \\ \mathcal{D}(F_c, F_c) \end{array} = \begin{array}{c} \boxed{id_{F_c}} \\ \downarrow \\ \mathcal{D}(F_c, F_c) \end{array}$$

1. Enrichment in virtual monoidal categories
2. Enrichment in virtual bicategories
3. Enrichment in virtual double categories
4. Lax slice enrichment
 - enrichment for restriction categories

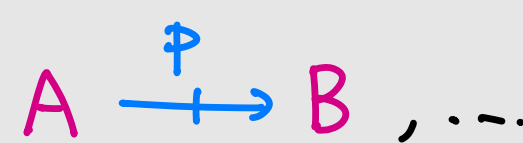
Virtual double categories

A virtual double category $\mathbb{V}$ consists of

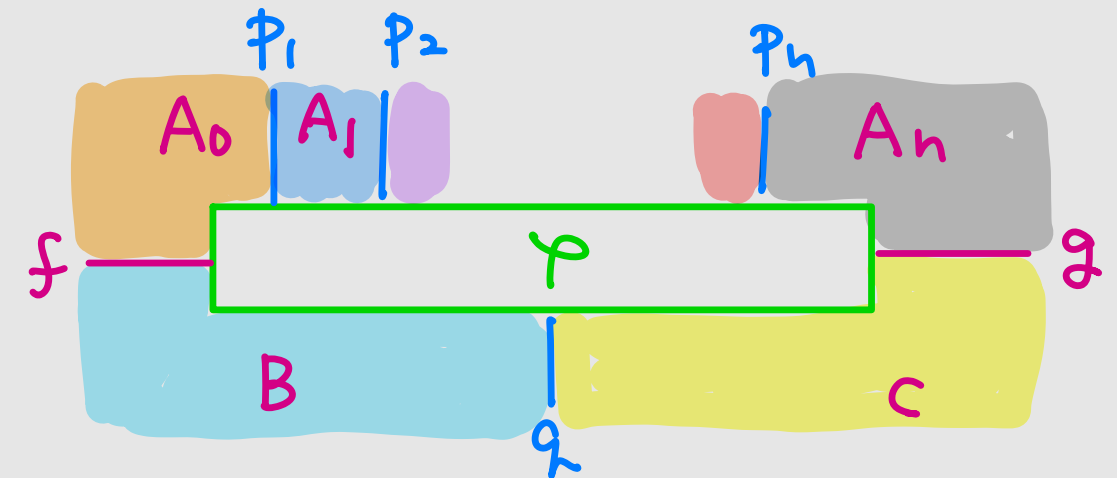
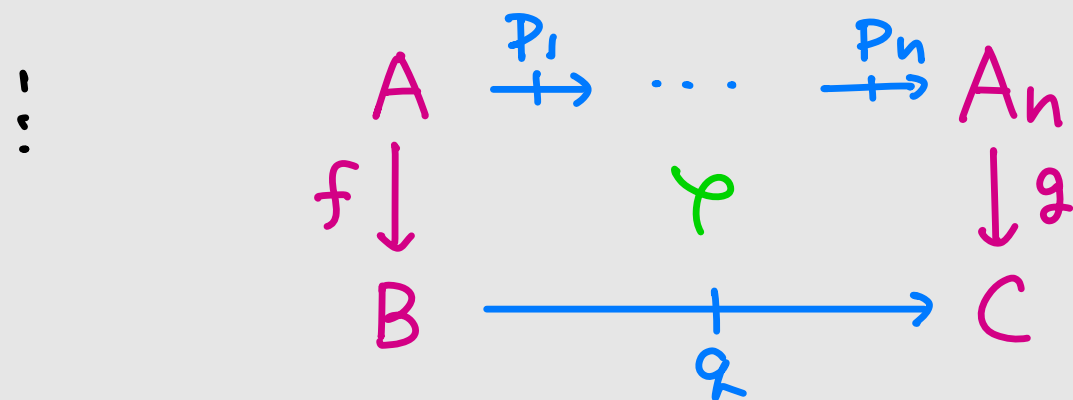
- a category $\mathbb{V}_0$ of **objects** and **tight morphisms**



- a collection of **loose morphisms**



- a collection of **multiary cells** ;

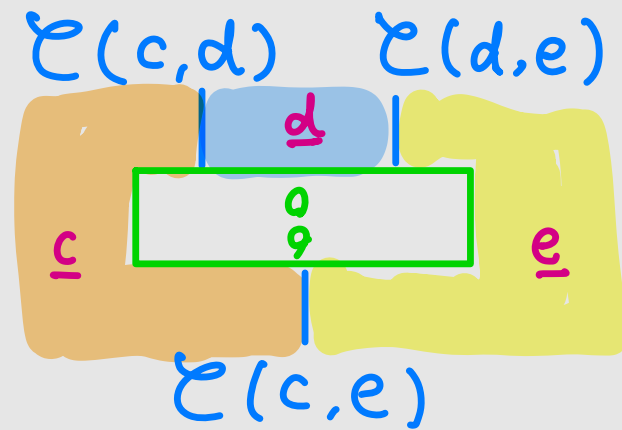


⚠ No composition of loose morphisms

Composition of cells is defined similarly to virtual bicategories.

Enrichment in a VDC

How should we use tight morphisms in the definition of categories?



No, thanks.

A virtual double category $\mathbb{V}$ has the underlying virtual bicategory $\mathcal{LV}$ of objects, loose morphisms, and cells with tight identities, and

$\mathbb{V}$ -categories = $\mathcal{LV}$ -categories.

A difference lies in functors.

Functors enriched in a VDC

A V-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- a function $|F|: |\mathcal{C}| \rightarrow |\mathcal{D}|$ with a family $\left(\begin{array}{c} \underline{c} \\ \hline \underline{F_c} \end{array} F_c \right)_{c \in |\mathcal{C}|}$;

$$\begin{array}{ccc} |\mathcal{C}| & \xrightarrow{|F|} & |\mathcal{D}| \\ \underline{(-)} \searrow & \Rightarrow & \swarrow \underline{(-)} \\ & \mathbb{V}_0 & \end{array}$$

- a family $\left(\begin{array}{c} \mathcal{C}(c, c') \\ \hline F_c \xrightarrow{\underline{c}} \boxed{F_{c,c'}} \xrightarrow{\underline{c'}} F_{c'} \\ \hline \underline{F_c} \quad \underline{F_{c'}} \\ \mathcal{D}(F_c, F_{c'}) \end{array} \right)_{c, c' \in |\mathcal{C}|}$;

s.t.

$$\begin{array}{ccc} \underline{c} & \xrightarrow{\mathcal{C}(c, c')} & \underline{c'} \\ F_c \downarrow & F_{c,c'} & \downarrow F_{c'} \\ \underline{F_c} & \xrightarrow{\mathcal{D}(F_c, F_{c'})} & \underline{F_{c'}} \end{array}$$

$$\begin{array}{c} \begin{array}{c} \underline{c'} \\ | \\ \boxed{id_{\mathcal{C}}} \\ | \\ \boxed{F_{c,c'}} \\ | \\ \underline{F_c} \end{array} \xrightarrow{\underline{c''}} \underline{F_{c''}} \\ \\ \end{array} = \begin{array}{c} \underline{c} \quad \underline{c'} \quad \underline{c''} \\ | \quad | \quad | \\ \boxed{F_{c,c'}} \quad \boxed{F_{c',c''}} \\ | \quad | \\ \underline{F_c} \quad \underline{F_{c'}} \quad \underline{F_{c''}} \\ | \\ \boxed{id_{\mathcal{D}}} \end{array} , \quad \begin{array}{c} \boxed{id_{\mathcal{C}}} \\ | \\ \boxed{F_{c,c}} \\ | \\ \underline{F_c} \end{array} \xrightarrow{\underline{F_c}} \underline{F_c} = \boxed{id_{F_c}}$$

Span vs \$pan

Remember ...

a Span-category = a $\text{\$pan-category}$ = a partitioned category
 $(\mathcal{X}, \mathcal{C}, \text{ob } \mathcal{X} \xrightarrow{F} \mathcal{C})$

$\uparrow$
 bicategory

$\uparrow$
 double category

$$\begin{array}{c}
 \text{\textcancel{Span}-functor} \\
 \text{Span}
 \end{array}
 \begin{array}{c}
 (\mathcal{X}, \mathcal{C}, F) \\
 \downarrow \\
 (\mathcal{Y}, \mathcal{D}, G)
 \end{array}
 \left| \left| \begin{array}{c}
 \left(\begin{array}{c} \mathcal{X} \\ \downarrow \Phi : \text{functor} \\ \mathcal{Y} \end{array}, \begin{array}{c} \mathcal{C} \\ \downarrow h : \text{function} \\ \mathcal{D} \end{array} \right) \\
 \text{s.t.} \quad \begin{array}{ccc} \text{ob } \mathcal{X} & \xrightarrow{F} & \mathcal{C} \\ \text{ob } \Phi \downarrow & \cong & \downarrow h \\ \text{ob } \mathcal{Y} & \xrightarrow{G} & \mathcal{D} \end{array} \quad \& \quad \Phi|_{F_c^{-1}} : F_c^{-1} \longrightarrow G^{-1}(h(c)) \\
 & & \text{is the identity}
 \end{array} \right.
 \end{array}$$

$\mathcal{Q}_L$ vs $\mathbb{G}_L$

L : a preordered set $\rightsquigarrow \mathcal{Q}_L$: a bicategory

objects : $a \in L$, $\mathcal{Q}_L(a, b) = \begin{cases} \{\perp \leq \top\} & \text{if } a \leq b, \\ \{\perp\} & \text{otherwise.} \end{cases}$

This extends to a double category $\mathbb{G}_L$ by $\begin{matrix} a \\ \downarrow \\ b \end{matrix} := \begin{matrix} a \\ \wedge \\ b \end{matrix}$.

$\mathcal{Q}_L$ -categories = $\mathbb{G}_L$ -categories

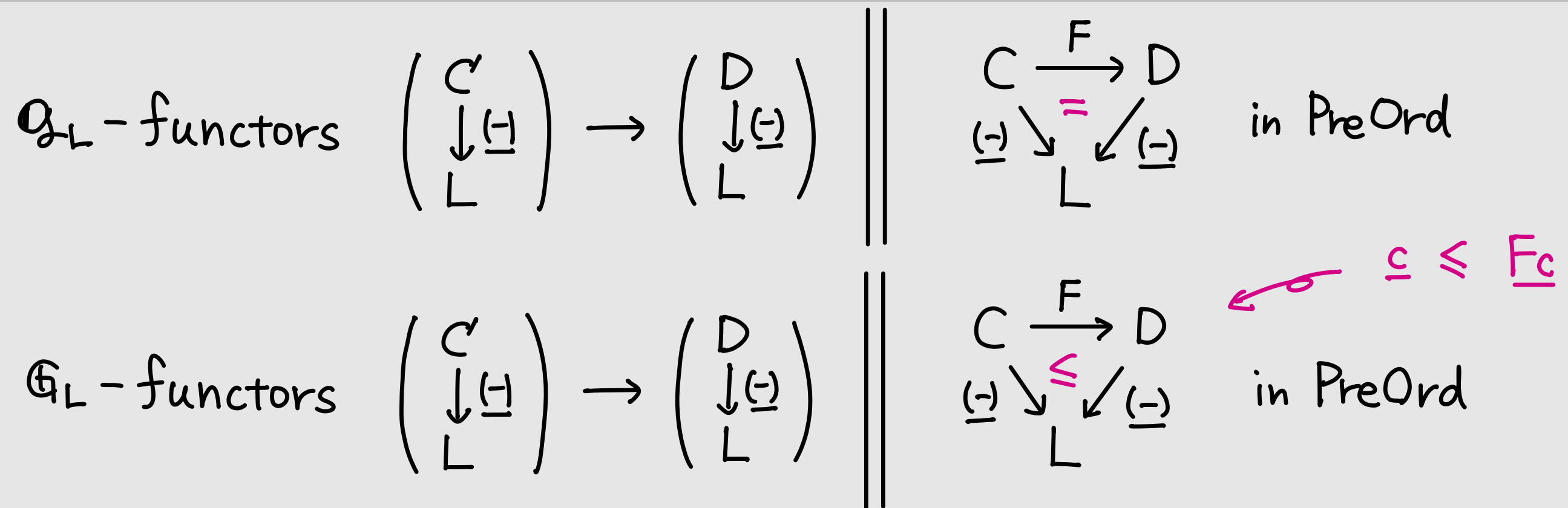
= pairs of $C \in \text{PreOrd}$ & $\begin{matrix} C \\ \downarrow (-) \\ L \end{matrix}$

$\mathcal{Q}_L$ vs $\mathbb{G}_L$

L : a preordered set $\rightsquigarrow \mathcal{Q}_L$: a bicategory

objects : $a \in L$, $\mathcal{Q}_L(a, b) = \begin{cases} \{\perp \leq \top\} & \text{if } a \leq b, \\ \{\perp\} & \text{otherwise.} \end{cases}$

This extends to a double category $\mathbb{G}_L$ by $\begin{matrix} a \\ \downarrow \\ b \end{matrix} := \begin{matrix} a \\ \wedge \\ b \end{matrix}$.



Restriction categories = $\mathbb{P}(\text{StabLat}^{\text{op}})$ -categories

Ever since categories enriched in a (virtual) double category was introduced by Leinster in "Generalized enrichment of categories" (2002), there has been **only one** paper that gives "naturally occurring examples", which is

Thm (Cockett, Garner "Restriction categories as enriched categories" (2014))

(the 2-category of restriction categories)

SI

(the 2-category of $\mathbb{P}(\text{StabLat}^{\text{op}})$ -categories)

Q. Where does this $\mathbb{P}(\text{StabLat}^{\text{op}})$ come from?

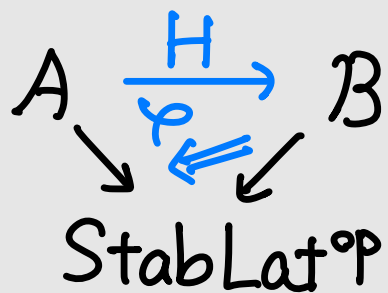
1. Enrichment in virtual monoidal categories
2. Enrichment in virtual bicategories
3. Enrichment in virtual double categories
4. Lax slice enrichment
 - enrichment for restriction categories

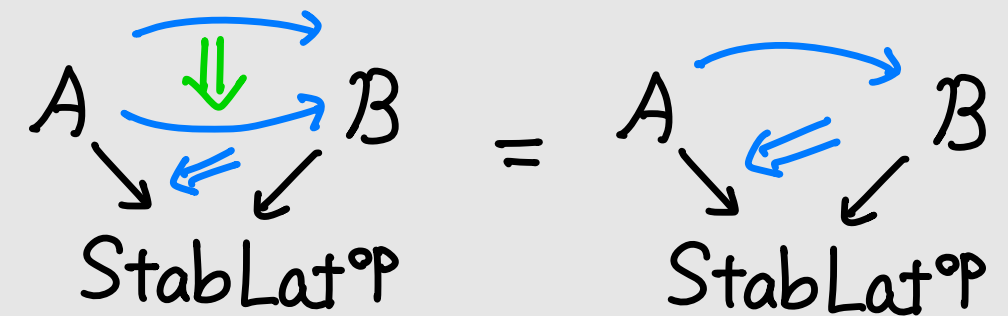
Outline of Cockett and Garner's result : Step 1

Prop The 2-category of restriction categories is equivalent to $2\text{-Cat} //_{\text{ldf}} \text{StabLat}^{\text{op}}$,
the locally-full sub-2-category of the lax slice $2\text{-Cat} // \text{StabLat}^{\text{op}}$

• 0-cells : $A \xrightarrow{F} \text{StabLat}^{\text{op}}$

• 2-cells :

• 1-cells : $A \xrightarrow{H} B$




whose

$\left\{ \begin{array}{l} \text{0-cells are local discrete fibrations, and} \\ \text{1-cells are those whose } \varphi_a \text{'s are representably discrete fibrations.} \end{array} \right.$

Outline of Cockett and Garner's result : Step 2

Thm For a 2-category $\mathcal{K}$, there is a double category $\mathbb{P}(\mathcal{K})$

s.t. $\mathbb{P}(\mathcal{K})\text{-Cat} \simeq 2\text{-Cat} //_{\text{ldf}} \mathcal{K}$.

- $\mathbb{P}(\mathcal{K})$:
- $\mathbb{P}(\mathcal{K})_0 = \mathcal{K}_0$
 - $\phi: x \rightarrow x'$ in $\mathbb{P}(\mathcal{K})$ is a presheaf on $\mathcal{K}(x, x')$.
 - Cells ... (use the convolution product)

Thm [Fuji, Lack "The oplax limit of an enriched category" (2024)]

$\mathcal{B}$: bicategory , $\mathcal{C}$: $\mathcal{B}$ -category $\rightsquigarrow \exists \mathcal{B}/\mathcal{C}$: bicategory

s.t. $(\mathcal{B}/\mathcal{C})\text{-Cat} \simeq \mathcal{B}\text{-Cat} / \mathcal{C}$.

Lax slice enrichment

Prop [ACCJNT]

$\mathbb{V} : \text{VDC}, \mathcal{C} : \mathbb{V}\text{-category} \rightsquigarrow \exists \mathbb{V} // \mathcal{C} : \text{VDC}$

s.t. $(\mathbb{V} // \mathcal{C})\text{-Cat} \simeq \mathbb{V}\text{-Cat} // \mathcal{C}$

$\mathbb{P}(\text{StabLat}^{\circ\mathbb{P}})$ is a sub-VDC of $\Sigma\text{Cat} // \text{StabLat}^{\circ\mathbb{P}}$,

where ΣCat is the cart.mon.cat. Cat seen as a double cat (the delooping).

Cor [ACCJNT]

$\mathbb{V} : \text{VDC}, \mathbb{V}' \subseteq \mathbb{V}, \mathcal{C} : \mathbb{V}\text{-category} \rightsquigarrow \exists \mathbb{V} //_{\mathbb{V}'} \mathcal{C} : \text{VDC}$

s.t. $(\mathbb{V} //_{\mathbb{V}'} \mathcal{C})\text{-Cat} \simeq \mathbb{V}\text{-Cat} //_{\mathbb{V}'} \mathcal{C}$

In fact, $\Sigma\text{Cat} //_{\Sigma\text{DFib}} \kappa$ is equivalent to $\mathbb{P}(\kappa)$.

Relators (a.k.a. distributors, profunctors, bimodules)

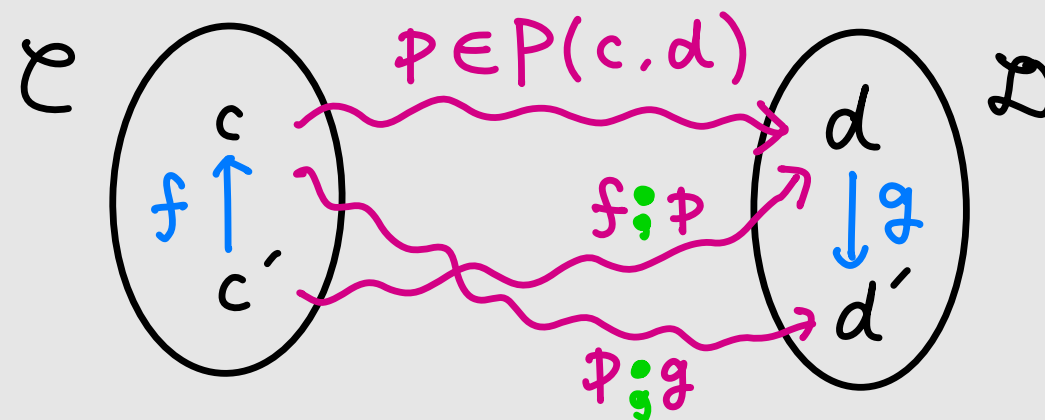
Def $\mathcal{C}, \mathcal{D}$: categories

A **relator** $\mathcal{C} \multimap \mathcal{D}$ is a family of sets $(P(c, d))_{c \in |\mathcal{C}|, d \in |\mathcal{D}|}$

with operations

$$\begin{array}{c} \mathcal{C}(c', c) \quad | \quad | \quad P(c, d) \\ \hline \lambda_P \\ \hline P(c', d) \end{array} \quad \& \quad \begin{array}{c} P(c, d) \quad | \quad | \quad \mathcal{D}(d, d') \\ \hline \rho_P \\ \hline P(c, d') \end{array}$$

that satisfy some axioms.



Categories :: Functors :: **Relators** $\approx$ Sets :: Functions :: Binary relations

VDC of categories, functors, and relators

$\mathbb{C}AT$: objects : **categories**, tight morphisms : **functors**

loose morphisms : **relators**, cells : **natural transformations**

This defines a virtual double category.

There are many variations of this :

- **$V\text{-}\mathbb{C}AT$** whose objects are V -categories.
- **$Rest\mathbb{C}AT$** whose objects are restriction categories
and loose morphisms are restriction relators .

DeWolff "Restriction category perspective of partial computation and geometry" (2017)

$$\underline{Thm} [ACCJNT] \quad (\Sigma Cat //_{\Sigma DFib} \mathcal{K})\text{-}\mathbb{C}AT \simeq Rest\mathbb{C}AT$$

$\mathbb{G}_L$ as a lax slice

Recall...

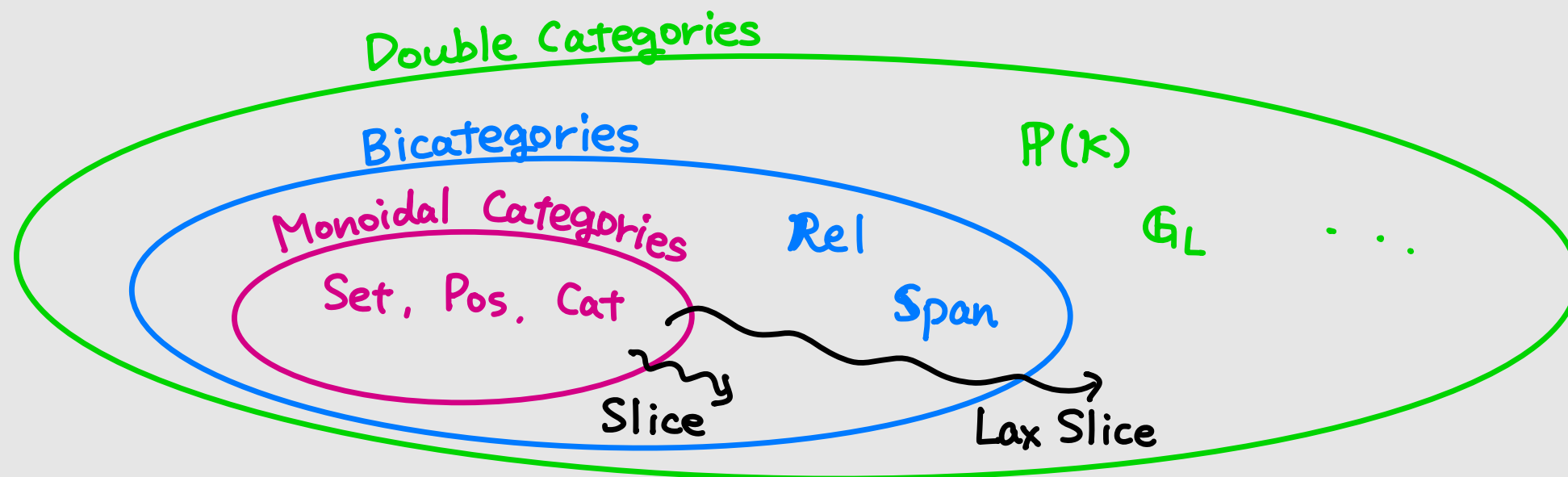
$$2 = \{\perp \leq \top\}$$

L : a preordered set $\rightsquigarrow \mathbb{G}_L$: a double category

$$(\mathbb{G}_L)_0 = L, \quad \mathbb{G}_L(a, b) = \begin{cases} \{\perp \leq \top\} & a \leq b \\ \{\perp\} & \text{otherwise} \end{cases} \cong 2 // L(a, b)$$

$$\mathbb{G}_L = 2 // L \rightsquigarrow \mathbb{G}_L\text{-CAT} = (2 // L)\text{-CAT} \simeq \text{Preord} // L$$

Remark This construction makes even a monoidal category into a double category.



Enrichment, functorially

- $(-)-\mathbb{C}AT$ gives a 2-functor $\mathcal{V}Db|Cat \longrightarrow \mathcal{N}\mathcal{V}Db|Cat$. the 2-cat. of
normal VDCs

In fact, it is the composite $\mathcal{V}Db|Cat \xrightarrow{(-)-Mat} \mathcal{V}Db|Cat \xrightarrow{Mod} \mathcal{N}\mathcal{V}Db|Cat$.

A $\mathcal{V}$ -category is (just) a monad in the virtual double category of $\mathcal{V}$ -matrices.

- $(-)-\mathbb{C}AT$ is a familial 2-functor. Fuji, Lack "The familial nature of enrichment
over virtual double categories" (2025)

In particular, it preserves all connected limits.

(This is not the case for $MonCat \xrightarrow{(-)-CAT} 2-Cat$ or $BiCat_l \xrightarrow{(-)-CAT} 2-Cat$.)

- The lax slice gives a 2-functor

$$* \downarrow (-)-\mathbb{C}AT \longrightarrow \mathcal{V}Db|Cat^{\rightarrow}; \quad (\mathcal{V}, \mathcal{E}) \mapsto \begin{array}{c} \mathcal{V} // \mathcal{E} \\ \downarrow \\ \mathcal{V} \end{array}$$

The equivalence we showed is 2-natural.

Future Work

- Find more examples and applications in restriction category theory (and beyond!)
- $(\mathcal{V}, \otimes, I)$: monoidal category
 $\leadsto (\hat{\mathcal{V}}, \otimes^{\text{conv}}, I^{\text{conv}})$: its presheaf category with (Day's) convolution product

$$(\hat{\mathcal{V}}, \otimes^{\text{conv}}, I^{\text{conv}})\text{-categories} = (\text{locally}) \mathcal{V}\text{-graded categories}$$

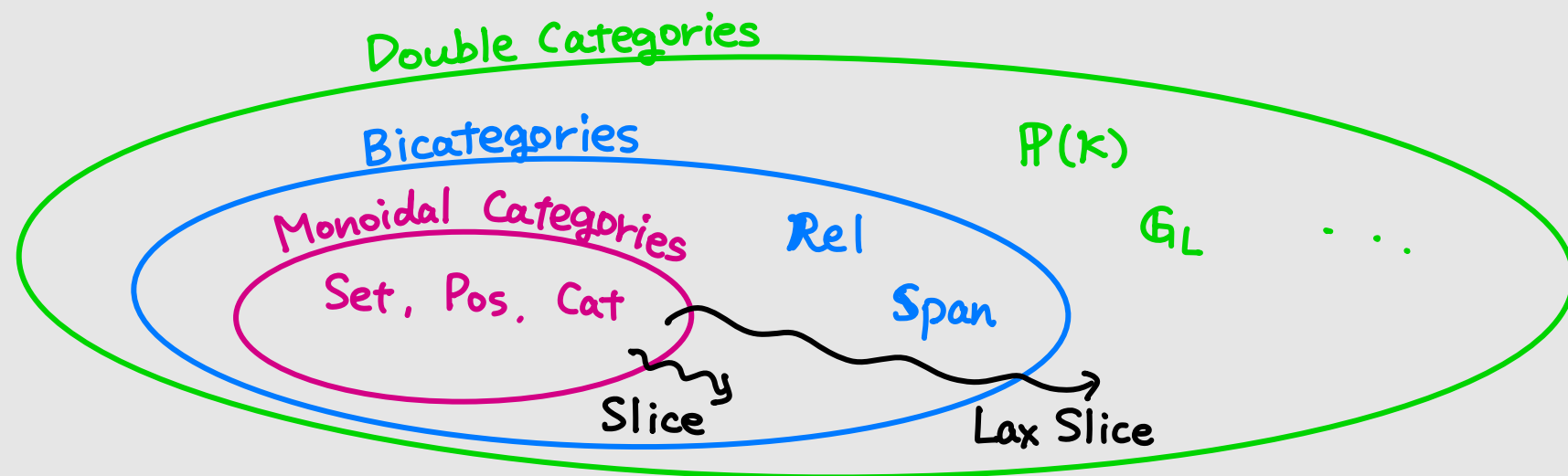
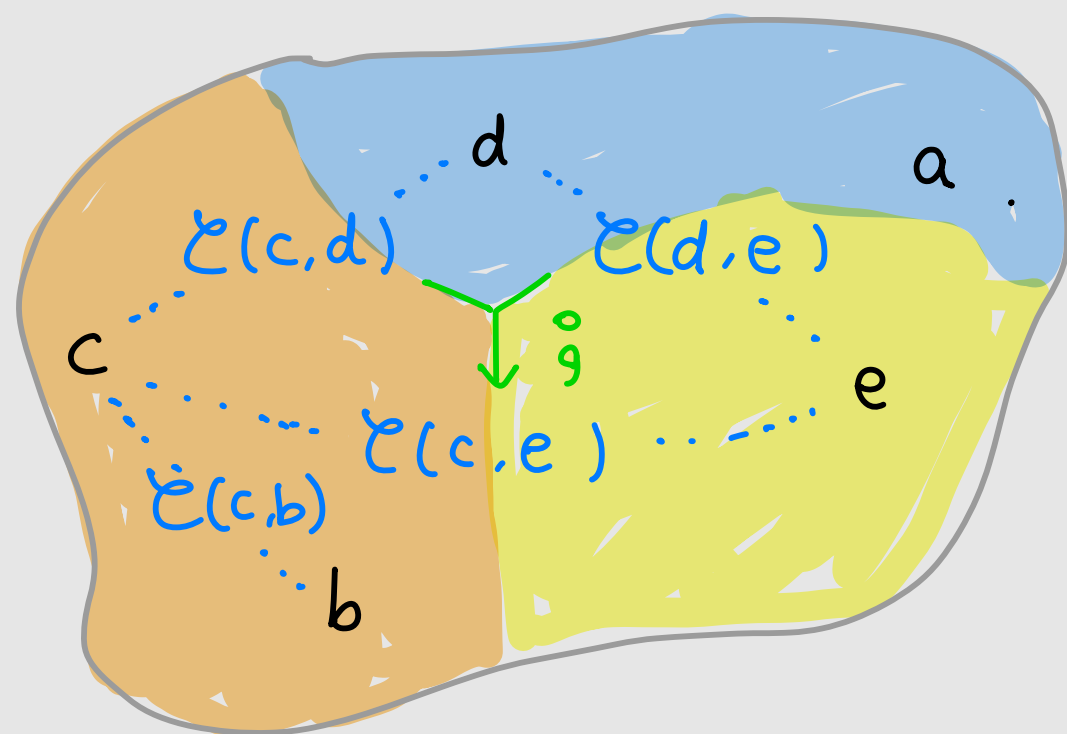
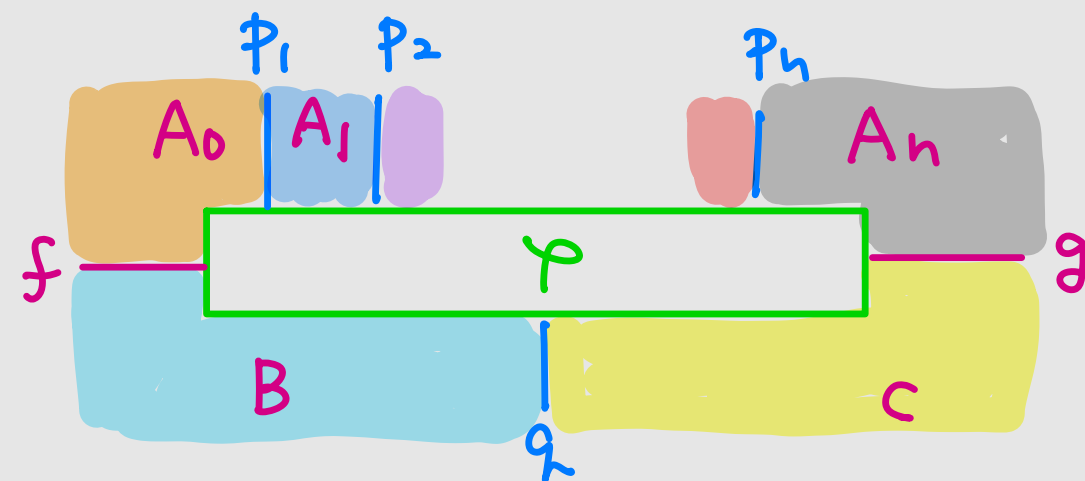
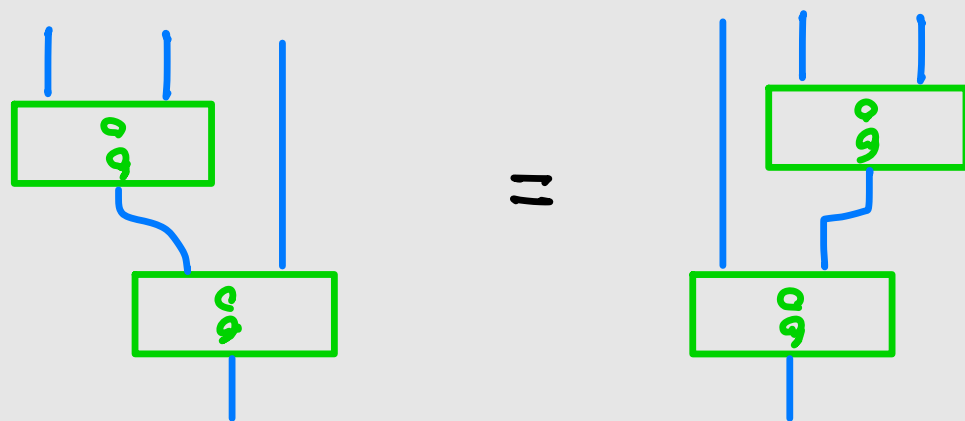
Wood "Indical methods for relative categories" (1976)

$\mathbb{P}(\mathcal{K})$ can be also seen as a generalization of this.

- Find a more abstract proof of the lax slice theorem.

$$\left(\begin{array}{l} B/\mathcal{E} \text{ is an oplax limit in } \mathbf{Bicat}_\ell, \text{ which is preserved by } (-)\text{-Cat}. \text{ [Fuji \& Lack, 2024]} \\ \text{However, } \mathcal{V} // \mathcal{E} \text{ is not a limit in } \mathbf{VDbCat}. \end{array} \right)$$

May your life be V-enriched!



Bonus : $\mathbb{V} // \mathcal{C}$

Objects : $\left(c \in |\mathcal{C}|, \begin{array}{c} x \\ \downarrow \zeta \\ \underline{c} \end{array} \text{ in } \mathbb{V} \right)$

Tight morphisms

$$\begin{array}{c} (c, x, \zeta) \\ \downarrow \\ (d, y, \zeta) \end{array} : \begin{array}{ccc} x & \xlongequal{\quad} & x \\ \downarrow \textcolor{violet}{f} & \textcolor{violet}{\varphi} & \downarrow \zeta \\ y & & d \\ \downarrow \zeta & \xrightarrow{\text{ } \mathcal{C}(c,d) \text{ } } & \underline{d} \\ \underline{c} & & \end{array}$$

Loose morphisms $(c, x, \zeta) \dashrightarrow (d, y, \zeta) : \begin{array}{ccc} x & \xrightarrow{\textcolor{violet}{l}} & y \\ \downarrow \zeta & \textcolor{violet}{p} & \downarrow \zeta \\ \underline{c} & \xrightarrow{\text{ } \mathcal{C}(c,d) \text{ } } & \underline{d} \end{array}$

Cells :

$$\begin{array}{ccccccc} x_0 & \xlongequal{\quad} & x_0 & \xrightarrow{\text{ } l_1 \text{ } } & x_1 & \dashrightarrow \dots \dashrightarrow & x_n \\ f_0 \downarrow & & \downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_n \\ y_0 & & \downarrow \zeta_0 & & \downarrow \zeta_1 & & \downarrow \zeta_n \\ \zeta_0 \downarrow & & & & & & \\ \underline{d_0} & \xrightarrow{\text{ } \mathcal{C}(d_0, c_0) \text{ } } & \underline{c_0} & \xrightarrow{\text{ } \mathcal{C}(c_0, c_1) \text{ } } & \underline{c_1} & \dashrightarrow \dots \dashrightarrow & \underline{c_n} \\ \parallel & & & & \text{0} & & \parallel \\ \underline{d_0} & \xrightarrow{\text{ } \mathcal{C}(d_0, c_n) \text{ } } & & & & & \underline{c_n} \end{array}$$

$$\begin{array}{ccccccc} x_0 & \xrightarrow{\text{ } l_1 \text{ } } & x_1 & \dashrightarrow \dots \dashrightarrow & x_n & \xlongequal{\quad} & x_n \\ f_0 \downarrow & & & & \downarrow f_1 & & \downarrow \zeta_n \\ y_0 & \xrightarrow{\text{ } k \text{ } } & y_1 & \xrightarrow{\text{ } \textcolor{violet}{\tau} \text{ } } & y_1 & \xrightarrow{\text{ } \varphi_1 \text{ } } & \downarrow \zeta_n \\ \zeta_0 \downarrow & & & & \downarrow \zeta_1 & & \\ \underline{d_0} & \xrightarrow{\text{ } \mathcal{C}(d_0, d_1) \text{ } } & \underline{d_1} & \xrightarrow{\text{ } \mathcal{C}(d_1, c_n) \text{ } } & \underline{c_n} & & \parallel \\ \parallel & & \text{0} & & & & \parallel \\ \underline{d_0} & \xrightarrow{\text{ } \mathcal{C}(d_0, c_n) \text{ } } & & & & & \underline{c_n} \end{array}$$