NOTES ON SEAR
(DRAFT)

HAYATO NASU

This is a private note on one of the structural set theories called SEAR (Sets, Elements, and Relations). It is originally introduced by Schulman in nLab [nLa23d]. The aim of this note is to rearrange the original definition of SEAR and to create another version of SEAR called SEFAR (Sets, Elements, Functions, and Relations). The main difference between SEAR and SEFAR is that SEFAR has a function symbol as a primitive symbol. SEFAR is akin to the structural ZFC [nLa23c] in this sense, but it still inherits the skeleton of SEAR, namely the tabulation of relations. Allowing a function symbol as a primitive symbol makes it possible to extend the theory to other settings like predicative constructive set theories, where the principle of unique choice is not provable. We also show that SEFAR is equivalently axiomatized in the language of double categories.

Having written almost all the contents of this note, I found that, in the page of SEAR+ε [nLa23a], he essentially introduces what I call SEFAR as a variant of SEAR to extend the theory to describe the Hilbert's ε operator. The important feature is that it does not have judgmental equality, but only the equivalence relations serving as the equality. Although it seems not to be completely developed in partucular in the predicative extension of this theory, this is what I wanted to construct. However, the double categorical aspect of SEFAR is not discussed in the page, so it still has some value to be written.

By first-order logic, we mean the constructive first-order logic unless otherwise stated.

1. Brief introduction of structural set theory

The origin of structural set theory is a monumantal paper [Law64] by Lawvere. The paper is a pioneering work in categorical logic, connecting set theory and category theory. It can be seen as a characterization of the category of sets, and as a noble foundation of mathematics via category theory.

The term "structural set theory" is coined by Schulman. The original idea of structural set theory is to axiomatize set theory with sets and functions as primitive notions, not with sets and the global membership relation. This perspective is also taken by many other authors, see [Lei14] for example. (It would be better for readers to understand the idea of structural set theory by consulting [Shu19, nLa23b] and other blog posts, [Mik09] for example, in the n-Category Café by Schulman.)

2. Supplementary material for double category theory

This article owes double category theory to the papers [HN23, Lam22]. but we will use additional definitions for our purpose.

Definition 2.1. Let $\mathbb{D}$ be a double category. $\mathbb{D}$ has ***right extensions*** if for any $p\colon A \nrightarrow B$ in $\mathbb{D}$, the functor

$$p \odot -\colon \mathbb{D}(B, C) \to \mathbb{D}(A, C)$$

has a right adjoint for any C in $\mathbb{D}$. We write $p\backslash(-)$ for the right adjoint of $p \odot -$. Similarly, $\mathbb{D}$ has ***right liftings*** if for any $p\colon A \nrightarrow B$ in $\mathbb{D}$, the functor

$$- \odot p\colon \mathbb{D}(C, A) \to \mathbb{D}(C, B)$$

has a right adjoint for any C in $\mathbb{D}$. We write $(-)/p$ for the right adjoint of $- \odot p$.

3. The axioms of SEAR

The language of SEAR. SEAR is based on dependent type theory with three sorts:

Date: December 7, 2023.

- Set is a sort, considered as the type of sets,
- $\mathsf{Element}(A)$ is a sort dependent on $A\colon \mathsf{Set}$, considered as the type of elements of A, and
- $\mathsf{Relation}(A, B)$ is a sort dependent on $A\colon \mathsf{Set}$ and $B\colon \mathsf{Set}$, considered as the type of relations from A to B.

For the sake of simplicity, we will write the type declarations $x\colon \mathsf{Element}(A)$ as $x \in A$, and the type declarations $p\colon \mathsf{Relation}(A, B)$ as $p\colon A \twoheadrightarrow B$ or $A \overset{p}{\twoheadrightarrow} B$.

The atomic formulas of SEAR consist of the following:

- For $A\colon \mathsf{Set}$ and $x, y \in A$, $x = y$ is an atomic formula.
- For $A, B\colon \mathsf{Set}$ and $p, q\colon A \twoheadrightarrow B$, $p = q$ is an atomic formula.
- For $A, B\colon \mathsf{Set}$ and $p\colon A \twoheadrightarrow B$, $p(x, y)$ is an atomic formula.

Formulas are constructed from atomic formulas using the connectives and quantifiers of first-order logic.

Remark 3.1. We do not have an equality symbol for sets. Since $x \in A$ is not a formula but a type declaration, we cannot even describe $A = B$ using the existing formula.

Relational comprehension and functions. We introduce two basic axioms of SEAR.

Axiom 0 (Nontriviality)**.** There exists $A\colon \mathsf{Set}$ such that $x \in A$ exists.

Axiom 1 (Relational comprehension)**.** For any $A, B\colon \mathsf{Set}$ and any first-order formula $\phi(x, y)$ with free variables $x \in A$ and $y \in B$, there exists a unique $p\colon A \twoheadrightarrow B$ such that $\phi(x, y) \leftrightarrow p(x, y)$ holds.

This is the axiom scheme, formulas $\phi(x, y)$ ranging over all first-order formulas with two free element-variables.

Definition 3.2. A relation $p\colon A \twoheadrightarrow B$ is ***functional*** if for any $x \in A$, there exists a unique $y \in B$ such that $p(x, y)$ holds. For a functional relation $p\colon A \twoheadrightarrow B$ and $x \in A$, we write $\widetilde{p}(x)$ for the unique element $y \in B$ such that $p(x, y)$ holds. From this viewpoint, we call $\widetilde{p}$ the ***function associated with*** p. We write a function as $f\colon A \to B$ instead of $f\colon A \twoheadrightarrow B$.

Remark 3.3. The notation $\widetilde{p}(x)$ is not adequate syntactically. However, since we allow the existencial quantifier in formulas, we can easily rewrite a formula into an equivalent one without the expression $\widetilde{p}(x)$.

Shulman treats functions in this way, but there can be another approach to define functions; we can set an additional dependent sort $\mathsf{Function}(A, B)$ from the outset, and a built-in function $\mathsf{ev}_{A,B}\colon \mathsf{Function}(A, B) \times A \to B$. We write $f(x)$ instead of $\mathsf{ev}_{A,B}(f, x)$. As atomic formulas about functions, we have $f = g$ where $f, g\colon A \to B$. One need to add the following axiom Axiom UC not present in original SEAR to get the equivalent foundation.

Axiom UC (Unique choice)**.** *For any $A, B\colon \mathsf{Set}$ and any functional relation $p\colon A \twoheadrightarrow B$, there exists a function $f\colon A \to B$ such that $p(x, y) \leftrightarrow f(x) = y$ holds.*

A careful observation reveals that the foundation consisting of Axioms 0 and 1 in SEAR is equivalent to the foundation consisting of Axioms 0 and 1 and axiom UC in the above approach.

Axiom AUC (Adjunctional Unique Choice). *For any* $A, B\colon \mathsf{Set}$ *and any* $p\colon A \twoheadrightarrow B$ *and* $q\colon B \twoheadrightarrow A$, *there exists a function* $f\colon A \to B$ *such that* $p(x,y) \leftrightarrow \mathsf{ev}(f,x) = y$ *and* $q(y,x) \leftrightarrow \mathsf{ev}(f,x) = y$ *hold if the following hold:*

$$\forall x \in A\, \exists y \in B\, (p(x,y) \wedge q(y,x)),$$
$$\forall x \in A, \forall y, y' \in B\, (q(y,x) \wedge p(x,y')) \to y = y'.$$

Proposition 3.4. *Under Axiom 1, Axiom UC is equivalent to Axiom AUC.*

Proof. We prove the implication from Axiom UC to Axiom AUC by showing that p in the statement of Axiom AUC is functional. The totality of p directly follows from the first formula of Axiom AUC. For the uniqueness, pick $b \in Y$ such that $p(x,b)$ and $q(b,x)$ hold for $x \in A$. If $y, y' \in B$ satisfy $p(x,y)$ and $p(x,y')$, respectively, then the second formula of Axiom AUC implies $y = b = y'$. Therefore, p is functional, and there exists a function $f\colon A \to B$ such that $p(x,y) \leftrightarrow \mathsf{ev}(f,x) = y$ holds. From the first formula, we have $y = f(x) \to q(x,y)$, and from the second formula, we have the converse.

For the implication from Axiom AUC to Axiom UC, one should take $q\colon B \twoheadrightarrow A$ to be the relation corresponding to the formula $y \in B, x \in A \vdash p(x,y)$ by Axiom 1. □

Proposition 3.5. *Under Axiom 1 and axiom UC, functions are extensional, i.e., for any* $A, B\colon \mathsf{Set}$ *and for any functions* $f, g\colon A \to B$, *if* $f(x) = g(x)$ *holds for any* $x \in A$, *then* $f = g$ *holds.*

Proof. Let f, g be functions associated with $p, q\colon A \twoheadrightarrow B$, respectively. We have the following:

$$\begin{aligned}
& A, B\colon \mathsf{Set}, p, q\colon A \twoheadrightarrow B, x \in A && \vdash \widetilde{p}(x) = \widetilde{q}(x) \\
\Leftrightarrow\ & A, B\colon \mathsf{Set}, p, q\colon A \twoheadrightarrow B, x \in A && \vdash \exists y \in B\, (p(x,y) \wedge q(x,y)) \\
\Rightarrow\ & A, B\colon \mathsf{Set}, p, q\colon A \twoheadrightarrow B, x, y \in A && \vdash p(x,y) \leftrightarrow q(x,y).
\end{aligned}$$

By Axiom 1, $p = q$ holds by uniqueness. □

If we drop Axiom UC, we have no way to define a new function from some given formula, nor to deduce the equality of functions from the equality of their values. However, it is reasonable to assume that we can compose functions in an effective way and functions are extensional.

This is the motivation of our formulation of SEFAR. It is an amalgamation of SEAR and ETCS with elements.

SEFAR

SEFAR is based on dependent type theory with the following sorts and built-in terms (families of terms):

- Set is a sort, considered as the type of sets,
- $\mathsf{Element}(A)$ is a sort dependent on $A\colon \mathsf{Set}$, considered as the type of elements of A, and
- $\mathsf{Function}(A,B)$ is a sort dependent on $A\colon \mathsf{Set}$ and $B\colon \mathsf{Set}$, considered as the type of functions from A to B.
- $\mathsf{Relation}(A,B)$ is a sort dependent on $A\colon \mathsf{Set}$ and $B\colon \mathsf{Set}$, considered as the type of relations from A to B.
- $\mathsf{ev}_{A,B}\colon \mathsf{Function}(A,B) \times A \to B$ is a built-in function dependent on $A, B\colon \mathsf{Set}$.
- $\mathsf{id}_A\colon \mathsf{Function}(A,A)$ is a built-in term dependent on $A\colon \mathsf{Set}$.
- $\mathsf{comp}_{A,B,C}\colon \mathsf{Function}(B,C) \times \mathsf{Function}(A,B) \to \mathsf{Function}(A,C)$ is a built-in function dependent on $A, B, C\colon \mathsf{Set}$.

As atomic formulas, we have the following:

- For $A\colon \mathsf{Set}$ and $x, y \in A$, $x = y$ is an atomic formula.
- For $A, B\colon \mathsf{Set}$ and $f, g\colon A \to B$, $f = g$ is an atomic formula.
- For $A, B\colon \mathsf{Set}$ and $p, q\colon A \twoheadrightarrow B$, $p = q$ is an atomic formula.

- For $A, B\colon \mathsf{Set}$ and $p\colon A \nrightarrow B$, $p(x, y)$ is an atomic formula.

For these three kinds of equalities, we assume the basic axioms of equality.

We continue to use the abbreviations we introduced above, and additionally, we write $f \circ g$ instead of $\mathsf{comp}_{A,B,C}(f, g)$.

Axiom FE (Function extensionality)**.** *Functions are extensional, i.e., for any $A, B\colon \mathsf{Set}$ and for any functions $f, g\colon A \to B$, $\forall x \in A\, f(x) = g(x)$ implies $f = g$.*

Axiom ECI (Evaluation of composition and identity)**.** *For any $A, B, C\colon \mathsf{Set}$ and for any $f\colon A \to B$ and $g\colon B \to C$, $\forall x \in A\, (g \circ f)(x) = g(f(x))$ and $\forall x \in A\, \mathsf{id}_A(x) = x$ hold.*

Even if we drop Axiom UC, it is natural to assume that we can define a new function from some given operation. This leads to the following axiom.

Axiom FO (Functions from operations)**.** *For any term judgment*

$$A, B\colon \mathsf{Set}, x \in A \vdash \varphi(x) \in B$$

where $\varphi(x)$ is a term of sort $\mathsf{Element}(B)$ dependent on $x \in A$, there exists a function $f\colon A \to B$ such that $\forall x \in A\, f(x) = \varphi(x)$ holds.

Terms $\varphi(x)$ range over all terms of sort $\mathsf{Element}(B)$ dependent on $x \in A$, so this is an axiom scheme. From now on, we assume Axioms 0 and 1 and axioms FE and ECI in SEFAR.

Lemma 3.6. *Composition of functions is associative and unital.*

Definition 3.7. For $A, B, C\colon \mathsf{Set}$ and $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$, we define $p \odot q\colon A \nrightarrow C$ by the formula $x \in A, z \in C \vdash \exists y \in Y\, (p(x, y) \wedge q(y, z))$ with help of Axiom 1. We call this relation the ***composite*** of p and q.

Lemma 3.8. *Composition of relations is associative and unital, where the unit is the equality relation $x \in A, x' \in A \vdash x = x'$.*

Proof. The statement follows from the basic axioms for equality including the Frobenius law of quantifiers as below:

$$x \in A \vdash \exists y \in B\, (\varphi(x, y) \wedge \psi(x)) \leftrightarrow (\exists y \in B\, \varphi(x, y)) \wedge \psi(x).$$

□

Definition 3.9. For a model $\mathcal{S}$ of SEFAR, we define a locally-preordered (strict) double category $\mathbb{R}\mathrm{el}_{\mathcal{S}}$ as follows:

- The objects are sets.
- The vertical arrows are functions.
- The horizontal arrows are relations.
- The cell of the form

$$\begin{array}{ccc} A & \overset{p}{\nrightarrow} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{q}{\nrightarrow} & D \end{array}$$

exists if and only if the following formula holds:

$$x \in A, y \in B \vdash p(x,y) \to q(f(x), g(y)).$$

There is at most one cell framed by given pairs of vertical and horizontal arrows.

We write $\mathcal{S}\mathrm{et}_{\mathcal{S}}$ for the vertical category of $\mathbb{R}\mathrm{el}_{\mathcal{S}}$, and $\mathcal{R}\mathrm{el}_{\mathcal{S}}$ for the horizontal bicategory of $\mathbb{R}\mathrm{el}_{\mathcal{S}}$.

It suffices to check that the composition of cells is well-defined. The vertical composition follows from the substitution of variables. For the horizontal composition, suppose the following formulas hold:

$$x \in A, y \in B \vdash p(x,y) \to p'(f(x), g(y)),$$
$$y \in B, z \in C \vdash q(y,z) \to q'(g(y), h(z)).$$

Then, we have the following:

$$\begin{aligned} & x \in A, z \in C \vdash && \exists y \in B\ (p(x,y) \wedge q(y,z)) \\ \Rightarrow\quad & x \in A, z \in C \vdash && \exists y \in B\ \big(p'(f(x), g(y)) \wedge q'(g(y), h(z))\big) \\ \Rightarrow\quad & x \in A, z \in C \vdash && \exists y' \in B\ \big(p'(f(x), y') \wedge q'(y', h(z))\big)\,. \end{aligned}$$

When we consider this meta-double category, we fix a model $\mathcal{S}$ of SEFAR and write $\mathbb{R}\mathrm{el}$ for $\mathbb{R}\mathrm{el}_{\mathcal{S}}$.

This double category is actually an equipment: for any niche

$$\begin{array}{ccc} A & & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{q}{\nrightarrow} & D \end{array}$$

in $\mathbb{R}\mathrm{el}$, we have the restriction $q(f,g)\colon A \nrightarrow B$ of q along f and g as

$$x \in A, y \in B \vdash q(f(x), g(y)).$$

In terms of this double category, Axiom FE states that $\mathbb{R}\mathrm{el}$ is unit-pure, while Axiom ECI is necessary for the vertical composition of the double category. Axiom AUC requires $\mathbb{R}\mathrm{el}$ to be Cauchy. Axiom 1 is so powerful that many consequences on the double category follow from it.

Tabulation. We introduce the third axiom of SEAR, which is not usually present in other set theories.

Axiom 2 (Tabulation)**.** For any $A, B\colon \mathsf{Set}$ and any $p\colon A \nrightarrow B$, there exist $C\colon \mathsf{Set}$, $l\colon C \to A$, and $r\colon C \to B$ such that

$$\forall x \in A, \forall y \in B\, p(x,y) \leftrightarrow \exists z \in C\, (l(z) = x \wedge r(z) = y),$$
$$\forall z, z' \in C\, (l(z) = l(z') \wedge r(z) = r(z')) \to z = z'.$$

Proposition 3.10. *For any $A, B\colon \mathsf{Set}$ and $p\colon A \nrightarrow B$, the set C in Axiom 2 is a strong tabulator of p in $\mathbb{R}\mathrm{el}$ in the sense of double category. In particular, such a C is unique up to isomorphism.*

Proof. Left to the reader. □

We call such a triple (C, l, r), or simply C, a ***tabulation*** of p. We add this axiom to SEFAR from now on.

From Axiom 2, we can define the cartesian structure of $\mathbb{R}\mathrm{el}$.

Proposition 3.11. *Under the axioms of SEFAR including Axiom 2, we have the following:*

(i) There exists a set 1 with a unique element. It gives the vertical terminal in $\mathbb{R}\mathrm{el}$.

(ii) For any $A, B\colon \mathsf{Set}$, there exists a set $A \times B$ with projections $\pi_0\colon A \times B \to A$ and $\pi_1\colon A \times B \to B$, which gives the vertical product of A and B in $\mathbb{R}\mathrm{el}$.

Therefore, $\mathbb{R}\mathrm{el}$ is a unit-pure cartesian equipment.

Proof. For (i), take a set A with an element a by Axiom 0. We define 1 to be the tabulation of the relation defined by $x \in A, y \in A \vdash x = a \wedge y = a$. The tabulation comes with $l : 1 \to A$ and $r : 1 \to A$ such that $z \in 1 \vdash l(z) = r(z) = a$, and for any $z, z' \in 1$, if $l(z) = l(z')$ and $r(z) = r(z')$, then $z = z'$ holds. Since the value of l and r is always a, we have $z = z'$ for any $z, z' \in 1$, and 1 is a set with a unique element. Since the equality relation on 1 is always true, 1 is the vertical terminal in $\mathbb{R}\mathrm{el}$.

For (ii), take a tabulation of the relation defined by $x \in A, y \in B \vdash x = x \wedge y = y$. Then, we have a set $A \times B$ with functions $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that

$$\begin{aligned} x \in A, y \in B \quad &\vdash \quad \exists z \in A \times B\,(\pi_0(z) = x \wedge \pi_1(z) = y), \\ z, z' \in A \times B \quad &\vdash \quad (\pi_0(z) = \pi_0(z') \wedge \pi_1(z) = \pi_1(z')) \to z = z'. \end{aligned}$$

This is equivalent to:

$$x \in A, y \in B \quad \vdash \quad \exists! z \in A \times B\,(\pi_0(z) = x \wedge \pi_1(z) = y).$$

It is easy to deduce the (2-dimensional) universal property of the product from this formula and Axiom FE. □

By Axiom FO, $x \in A$ corresponds to a function $f : 1 \to A$, which leads to the fact that $\mathcal{S}\mathrm{et}$ is well-pointed.

We define some basic classes of functions and relations. Let us write $p^\circ : B \nrightarrow A$ for the relation defined by $y \in B, x \in A \vdash p(x, y)$ for $p : A \nrightarrow B$.

Definition 3.12. For $A, B : \mathsf{Set}$ and take $f : A \to B$ and $p : A \nrightarrow B$.

(i) f is ***injective*** if $\forall x, x' \in A\, f(x) = f(x') \to x = x'$ holds.
(ii) f is ***surjective*** if $\forall y \in B\, \exists x \in A\ f(x) = y$ holds.
(iii) p is ***bijective*** if p and p° are functional.

Proposition 3.13. *Under the axioms of SEFAR,*

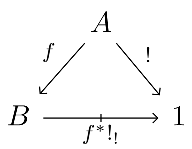
(i) f is injective if and only if it is an inclusion in $\mathbb{R}\mathrm{el}$.
(ii) f is surjective if and only if it is a final arrow in $\mathbb{R}\mathrm{el}$.

Proof. First, we prove (i). A vertical arrow $f : A \to B$ is an inclusion if and only if the identity cell

$$\begin{array}{ccc} A & \nrightarrow & A \\ f\downarrow & = & \downarrow f \\ B & \nrightarrow & B \end{array}$$

is a cartesian cell. Since $\mathbb{R}\mathrm{el}$ is locally preordered, this is equivalent to the existence of the horizontal cell from $\mathtt{Id}_B(f, f)$ to $\mathtt{Id}_A$.

Next, we prove (ii). A vertical arrow $f : A \to B$ is final if and only if the bottom horizontal arrow of the following opcartesian cell is a terminal in the horizontal hom-category:



In $\mathbb{R}\mathrm{el}$, $f^*!_!$ is the relation defined by

$$\begin{aligned} & y \in B, z \in 1 && \vdash \quad \exists x \in A\,(f(x) = y \wedge !(x) = z), \\ \Leftrightarrow \quad & y \in B, z \in 1 && \vdash \quad \exists x \in A\,(f(x) = y). \end{aligned}$$

Therefore, f is final if and only if f is surjective. □

We would like to prove that $\mathbb{R}\mathrm{el}$ is a unit-pure double category of relations in the sense of DCR. We can detect the fibrations in $\mathbb{R}\mathrm{el}$ as follows: A vertical arrow $f : A \to B$ is a fibration if and only if there exists a relation $p : B \nrightarrow 1$ such that

$$y \in B, z \in 1 \quad \vdash \quad (\exists x \in A\, f(x) = y) \leftrightarrow p(y, z).$$

This corresponds to the notion of *subset* defined in the nLab page of SEAR.

Lemma 3.14. *Under the axioms of SEFAR, fibrations in* $\mathbb{R}\mathrm{el}$ *are closed under composition.*

Proof. Let $f\colon A \to B$ and $g\colon B \to C$ be fibrations. Then, there exist relations $p\colon B \nrightarrow 1$ and $q\colon C \nrightarrow 1$ such that

$$\begin{aligned} y \in B, z \in 1 \quad &\vdash \quad (\exists x \in A\ f(x) = y) \leftrightarrow p(y, z), \\ w \in C, z \in 1 \quad &\vdash \quad (\exists y \in B\ g(y) = w) \leftrightarrow q(w, z). \end{aligned}$$

Define the relation $r\colon C \nrightarrow 1$ by

$$w \in C, z \in 1 \quad \vdash \quad (\exists y \in B\ (p(y, z) \wedge g(y) = w)).$$

Then, we can show the following:

$$w \in C, z \in 1 \quad \vdash \quad (\exists x \in A\ (g \circ f)(x) = w) \leftrightarrow r(w, z).$$

□

Theorem 3.15. $\mathbb{R}\mathrm{el}_{\mathcal{S}}$ *is a locally posetal double category of relations for any model* $\mathcal{S}$ *of SEFAR.*

From this, one can deduce that $\mathbb{R}\mathrm{el}$ satisfies the modular law, thus, $\mathcal{R}\mathrm{el}$ is an allegory.

Furthermore, we can prove that $\mathcal{R}\mathrm{el}$ is a division allegory using the universal quantifiers. This means that $\mathbb{R}\mathrm{el}$ has all right extensions and right liftings.

Proposition 3.16. $\mathbb{R}\mathrm{el}$ *has all right extensions and right liftings.*

Proof. The right extension of $q\colon A \nrightarrow C$ along $p\colon A \nrightarrow B$ is attained by the following formula:

$$y \in B, z \in C \quad \vdash \quad \forall x \in A\, (p(x, y) \to q(x, z)).$$

The existence of right liftings is proved similarly, or just by the involutive property of $\mathbb{R}\mathrm{el}$. □

Power sets. We introduce another axiom of SEFAR.

Axiom 3 (Power set)**.** For any $A\colon \mathsf{Set}$, there exist a set $P(A)$ and a relation $\varepsilon_A\colon A \nrightarrow P(A)$ and a dependent term

$$p\colon A \nrightarrow 1 \vdash \gamma(p) \in P(A)$$

such that

$$\forall x \in A, \forall p\colon A \nrightarrow 1, \forall y \in P(A)\ (y = \gamma(p) \leftrightarrow (p(x) \leftrightarrow \varepsilon_A(x, y)))\,.$$

Proposition 3.17. $\mathbb{R}\mathrm{el}$ *has power objects. That is, for any object* A*, the functor* $\mathcal{R}\mathrm{el}(A, -)$ *from* $\mathcal{S}\mathrm{et}^{\mathsf{op}}$ *to* **Set** *defined by restriction has a representing object.*

Proof. Let A be an object of $\mathbb{R}\mathrm{el}$. We show that $\varepsilon_A\colon A \nrightarrow P(A)$ is a universal element of $\mathcal{R}\mathrm{el}(A, -)$. Let B be an object of $\mathbb{R}\mathrm{el}$ and $p\colon A \nrightarrow B$ be a relation. Then, we have the following term-judgment:

$$y \in B \vdash \gamma\,(p(-, y)) \in P(A).$$

This gives a function $f\colon B \to P(A)$ such that

$$x \in A, y \in B \vdash p(x, y) \leftrightarrow \varepsilon_A(x, f(y)).$$

The uniqueness of f follows from the extensionality of functions and the uniqueness of $\gamma(-)$. □

In particular, taking the identity relation $\mathtt{Id}_A\colon A \nrightarrow A$, we get the function $\{-\}\colon A \to P(A)$. Moreover, we can define $\emptyset_A \in P(A)$ by $\bot\colon A \nrightarrow 1$. By tabulating the relation from $P(A)$ to $P(B)$ corresponding to the formula

$$x \in P(A), y \in P(B) \vdash \exists a \in A\, (x = \{a\} \wedge y = \emptyset_B) \vee \exists b \in B\, (x = \emptyset_A \wedge y = \{b\}),$$

we can define the disjoint coproduct $A \sqcup B$ of A and B.

If we assume the unique choice axiom Axiom UC, we can construct the exponentials in **Set** from the power objects since we can define the condition of being a function internally. However, we do not assume Axiom UC in SEFAR, there is no expectation that we can construct the exponentials in **Set** from the power objects. Therefore, attention should be paid to a pair of alternative axioms of SEFAR.

Axiom EXP (Exponentials)**.** *For any $A, B\colon \mathsf{Set}$, there exists a set B^A such that a function of form $A \to B$ corresponds to an element of B^A.*

Axiom SC (Subset classifier)**.** *There exists a set Ω and a relation $\tau\colon \Omega \nrightarrow 1$ and a dependent term*

$$p\colon A \nrightarrow 1 \vdash \gamma(p)\colon A \to \Omega$$

such that

$$\forall p\colon A \nrightarrow 1, \forall f\colon A \to \Omega, \forall x \in A, \left(f(x) = \gamma(p)(x) \leftrightarrow (p(x) \leftrightarrow \tau(f(x)))\right).$$

Proposition 3.18. *(i) If we assume Axioms EXP and SC,then Axiom 3 holds.*
(ii) If we assume Axioms 0 to 3, then Axiom SC holds.
(iii) If we assume Axioms 0 to 3 and Axiom UC, then Axiom EXP holds.

Mimicking the proof in [Joh02, Remark A2.4.10], we can prove the following:

Proposition 3.19. *Under the axioms of SEFAR, $\mathbb{R}\mathrm{el}$ has partial function representers, where a partial function is a relation $p\colon A \nrightarrow B$ such that $pp^\circ \leq \mathtt{Id}_A$.*

According to [Wyl91, §19], a finitely complete category with a proper stable factorization system (E, M) is a quasitopos, if and only if it has a M-partial function representers and is cartesian closed. Therefore we have the following:

Theorem 3.20. *Under the axioms of SEFAR Axioms 0 to 3 with Axiom EXP, $\mathbb{R}\mathrm{el}$ is a double category of strong relations on a quasitopos. Hence, if we assume Axiom UC, then $\mathbb{R}\mathrm{el}$ is a double category of relations on a topos.*

We can also define the quotient of an equivalence relation $p\colon A \nrightarrow A$.

Lemma 3.21. *For any $A\colon \mathsf{Set}$ and any equivalence relation $p\colon A \nrightarrow A$, there exists a set A/p and a surjection $g\colon A \to A/p$ such that*

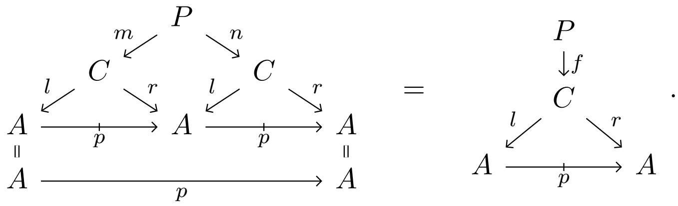
$$x, x' \in A \vdash g(x) = g(x') \leftrightarrow p(x, x').$$

Proof. See [nLa23d, Theorem 2.14]. □

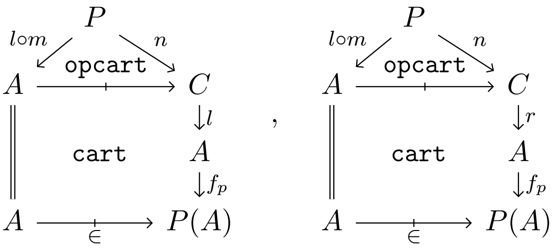
On the other hand, the universal property of quotients is not provable in SEFAR without Axiom UC. This suits our intuition that Cauchy double categories are counterparts of regular categories. However, we can prove the following:

Proposition 3.22. *For a locally-posetal double category of relations $\mathbb{D}$ with power objects, every symmetric monoid $p\colon A \nrightarrow A$ is a kernel of the corresponding vertical arrow $f_p\colon A \to P(A)$.*

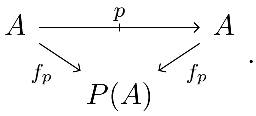
Proof. There must be a better proof via double category. Let $p\colon A \nrightarrow A$ be a symmetric monoid. Take a tabulator C of p with $l\colon C \to A$ and $r\colon C \to A$. The symmetricity of p implies that there exists an isomorphism $i\colon C \to C$ such that $l \circ i = r$ and $r \circ i = l$. The transitivity of p then implies that the pullback P of r and l factors through $\langle l, r\rangle\colon C \to A \times A$ as follows:



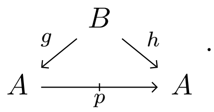
By the diagrammatic reasoning in [HN23] shows that the extension of $l \circ m$ along n is a restriction of p along l, and also a restriction of p along r.



Therefore, $f_p \circ l = f_p \circ r$ holds, and there exists a cell



If $f_p \circ g = f_p \circ h$ holds for some $g, h\colon B \to A$, then the restriction of p along $\mathtt{id}_A$ and h is equal to the restriction of $\in$ along $\mathtt{id}_A$ and $f_p \circ h = f_p \circ g$, so we have $g^* \subseteq p(\mathtt{id}_A, h)$. This leads to a cell



Thus, p is a kernel of f_p. □

From this observation, it may or may not be reasonable to assume another axiom of SEFAR: that is the existence of quotients of kernel relations.

Infinity. The axiom of infinity in SEAR is the following:

> **Axiom 4** (Infinity)**.** There exists a set N, an element o, and a function $s\colon N \to N$ such that
> $$\forall x, y \in N\, s(x) = s(y) \to x = y,$$
> $$\forall x \in N\, s(x) \neq o.$$

Collection. By an A-indexed family of sets, we mean a relation $p\colon X \nrightarrow A$ from a set X. Let p_a denote the tabulation of the restriction of p along $a\colon 1 \to A$. Then, p_a is a subset of X.

> **Axiom 5** (Collection)**.** For any $A\colon \mathsf{Set}$ and any first-order formula $\phi(x, X)$ with free variable $x \in A$, and a set-variables X, there exists sets B, a function $g\colon B \to A$, and a B-indexed family of sets $p\colon Y \nrightarrow B$ such that
> (i) $\forall b \in B\, \phi(g(b), p_b)$ holds, and
> (ii) $\forall a \in A, \forall X\colon \mathsf{Set}\ (\phi(a, X) \to \exists b \in B\, a = g(b))$ holds.

It is observed in the nLab page of SEAR that these axioms Axioms 0 to 5 in SEAR have equivalent strength to the axioms of ZF. Therefore, in SEFAR, we can prove that the set of axioms Axioms 0 to 5 and axioms FE to FO with Axiom UC is equivalent to the axioms of ZF.

4. Prospects

Dropping the axiom of unique choice can open up a possibility of Bishop-style constructive set theory, in SEFAR. The foundation without unique choice is investigated by Maietti and Sambin as the ***minimalist foundation***. The minimalist foundation, devised by Maietti [Mai09], is a foundation with two different levels: the extensional level and the intensional level. The interaciton between the two levels is conducted by the setoid construction of the extensional level by the intensional level. Intuitively, the extensional level is a stage of mathematics, and it seems natural to me that the level be presented as a double category.

References

[HN23] Keisuke Hoshino and Hayato Nasu. Double categories of relations relative to factorisation systems, 2023.

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume 1. Oxford University Press, 2002.

[Lam22] Michael Lambert. Double categories of relations. *Theory Appl. Categ.*, 38:Paper No. 33, 1249–1283, 2022.

[Law64] F. William Lawvere. An elementary theory of the category of sets. *Proc. Nat. Acad. Sci. U.S.A.*, 52:1506–1511, 1964.

[Lei14] Tom Leinster. Rethinking set theory. *The American Mathematical Monthly*, 121(5):pp. 403–415, 2014.

[Mai09] Maria Emilia Maietti. A minimalist two-level foundation for constructive mathematics. *Annals of Pure and Applied Logic*, 160(3):319–354, 2009.

[Mik09] Mike Shulman. Syntax, semantics, and structuralism 2. blog post in n-Category Café, December 2009.

[nLa23a] nLab authors. SEAR plus epsilon. `https://ncatlab.org/nlab/show/SEAR+plus+epsilon`, December 2023. Revision 13.

[nLa23b] nLab authors. structural set theory. `https://ncatlab.org/nlab/show/structural+set+theory`, November 2023. Revision 37.

[nLa23c] nLab authors. structural ZFC. `https://ncatlab.org/nlab/show/structural+ZFC`, December 2023. Revision 14.

[nLa23d] nLab authors (mainly Mike Shulman according to the history). SEAR. `https://ncatlab.org/nlab/show/SEAR`, November 2023. Revision 66.

[Shu19] Michael Shulman. Comparing material and structural set theories. *Ann. Pure Appl. Logic*, 170(4):465–504, 2019.

[Wyl91] Oswald Wyler. *Lecture Notes on Topoi and Quasitopoi*. World Scientific, 1991.

Email address: `hnasu@kurims.kyoto-u.ac.jp`

Research Institute of Mathematical Science, Kyoto University