Terminals are Pointwise

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Abstract

Limits of diagrams in a functor category $[\mathbf{I}, \mathbf{C}]$ are not necessarily pointwise when the category $\mathbf{C}$ is not complete. We show that no counterexample exists for terminal objects, or more generally, for limits of diagrams that takes values in the constant functors.

I would like to thank Yuki Maehara and Vincent Moreau for discussions and comments. I also regret that I could not provide a full proof of the main theorem, excusing myself by saying that I am currently busy with my master's thesis.

Notation

$\mathbf{C}$	a category
$\mathtt{id}_A$	the identity morphism of an object A
$\mathbf{n}$	the category with n objects and a unique morphism from i to j for all $i \leqslant j$
$[\mathbf{I}, \mathbf{C}]$	the category of functors from $\mathbf{I}$ to $\mathbf{C}$
$\Delta_{\mathbf{I}}$	the diagonal functor from $\mathbf{C}$ to $[\mathbf{I}, \mathbf{C}]$ that sends an object A to the constant functor at A

Are limits in functor categories necessarily pointwise?

One of the most famous exercises in category theory is to show that limits of diagrams in a functor category $[\mathbf{I}, \mathbf{C}]$ exist pointwise when the category $\mathbf{C}$ is complete. For example, see the exercise in [Rie16, Section 3.4][1]. Having learned this, a student with keen eyes may wonder if this is still true when the category $\mathbf{C}$ is not complete.

The answer is no.

Example 1 (Pullback)**.** This example is taken from [Kel05, Example 3.3]. Consider the category $\mathbf{C}$ generated by the following diagram:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \text{ where } h \circ f = h \circ g \text{ but } f \neq g$$

However, in the arrow category $[\mathbf{2}, \mathbf{C}]$, we have the following pullback diagram:

$$\begin{array}{ccc} g & \xrightarrow{(\mathtt{id}_A, \mathtt{id}_B)} & g \\ {\scriptstyle (\mathtt{id}_A, \mathtt{id}_B)} \downarrow & & \downarrow {\scriptstyle (f,h)} \\ g & \xrightarrow[(f,h)]{} & h \end{array}$$

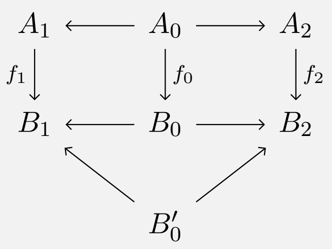
This is equivalent to saying that (f, h) is a monomorphism, which indeed is the case because only possible morphism into g in $[\mathbf{2}, \mathbf{C}]$ is $\mathtt{id}_A \xrightarrow{(\mathtt{id}_A, g)} g$ and the identity on the codomain g.

[1] I seem to have typeset my answer (https://drive.google.com/file/d/12Tth38Vjb2eeGQ_7Sy1MJtxVepHR_bON/view) when I was an undergrad.

Looking at the codomain part, however, we see that this is not a pointwise pullback, as h is not a monomorphism.
This does not contradict the theorem presented in the beginning, as the category $\mathbf{C}$ does not have the pullback of h and itself.

There are other counterexamples.

Example 2 (Binary product)**.** We can also consider the category $\mathbf{C}$ generated by the following diagram:



with the two squares commutative. In the arrow category $[\mathbf{2}, \mathbf{C}]$, the product of f_1 and f_2 is given by f_0, while the product of B_1 and B_2 is not B_0. This gives an example of non-pointwise product.
Again, the product of B_1 and B_2 does not exist in $\mathbf{C}$.

Exercise 3. Find a non-pointwise equalizer in $[\mathbf{2}, \mathbf{C}]$ for some category $\mathbf{C}$, or other functor categories. I observed there is one constructed similarly to the above examples.

Are these the most simple non-pointwise limits?

OK, so we have seen that limits in functor categories are not necessarily pointwise. But are the above diagrams the most simple examples? Those diagrams (pullback, binary product) are still quite small, but if we seek for more simple examples, what should we look for? It should be:

non-pointwise terminal objects.

No such thing exists!

The bad or good news is that we never have a non-pointwise terminal object in a functor category.

Theorem 4. *Let $\mathbf{I}$ be a small category and $\mathbf{C}$ be a category. If there is a terminal object in $[\mathbf{I}, \mathbf{C}]$, then it is pointwise. In particular, if $\mathbf{I}$ is not empty, a terminal object in $\mathbf{C}$ exists, and hence, the constant functor at the terminal object is isomorphic to the terminal object in $[\mathbf{I}, \mathbf{C}]$.*
Thus, the diagonal functor $\Delta_{\mathbf{I}} \colon \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$ creates terminal objects.

A proof of this statement was given by Yuki Maehara. The proof is not as simple as one may expect. I am not going to write the proof here; instead I will give a more general statement and its proof[2].

Theorem 5. *Let $\mathbf{I}, \mathbf{K}$ be small categories and $\mathbf{C}$ be a category. For any diagram $D \colon \mathbf{K} \to \mathbf{C}$, if a limit F of the diagram $\Delta_{\mathbf{I}} \circ D \colon \mathbf{K} \to [\mathbf{I}, \mathbf{C}]$ exists, then it is pointwise.*

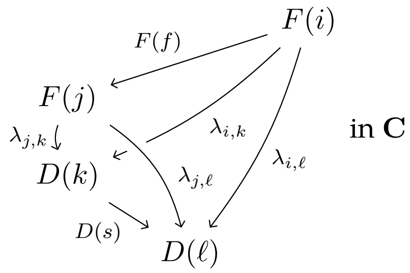
[2]Please let me know if this is already known somewhere.

Thus, if $\mathbf{I}$ is not empty, the diagonal functor $\Delta_{\mathbf{I}}\colon \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$ creates limits, even when $\mathbf{C}$ is not complete.

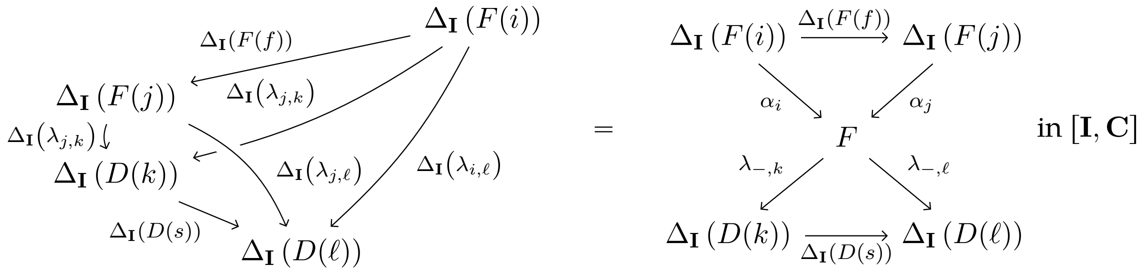
Theorem 4 follows from this theorem, because the diagram of a terminal object is empty, which trivially factors through the diagonal functor Δ.

Sketch of proof. I will illustrate how the proof works by extracting a general pair of arrows $f\colon i \to j$ in $\mathbf{I}$ and $s\colon k \to \ell$ in $\mathbf{K}$.

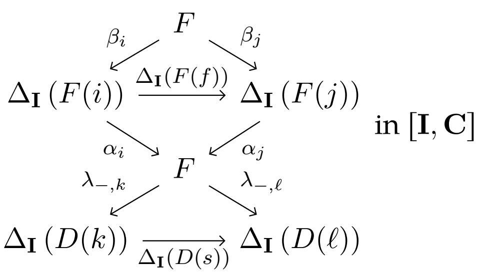
A limit cone λ of $F \in [\mathbf{I}, \mathbf{C}]$ is a natural transformation from the constant functor $\Delta_{\mathbf{K}}(F)\colon \mathbf{K} \to [\mathbf{I}, \mathbf{C}]$ to a diagram $\Delta_{\mathbf{I}} \circ D\colon \mathbf{K} \to [\mathbf{I}, \mathbf{C}]$. Unpacking the data, we obtain the following tetrahedron:



(It might be helpful to think of the diagram as a tetrahedron in a 3D space, where D lies on the ground and F floats above it.) The diagonal functor $\Delta_{\mathbf{I}}$ sends this tetrahedron to a tetrahedron in $[\mathbf{I}, \mathbf{C}]$, but here we know the diagram on the ground has a limit F, so the diagram breaks down to two triangles:



Now we have an i-indexed family of i'-indexed families of arrows $\left(\alpha_{i,i'}\colon F(i) \to F(i')\right)_{i' \in \mathbf{I}}$ natural in i', which is also natural in i by the universal property of the limit. A trick is to switch the order of taking indices: we can regard α as an i'-indexed family of i-indexed families, which gives us natural transformations from F to $\Delta_{\mathbf{I}}(F(i'))$'s. Let us write this as $\beta_i\colon F \to \Delta_{\mathbf{I}}(F(i))$ to avoid confusion. Then, it fits into the following diagram:



By the universal property of the limit, we have $\alpha_i \circ \beta_i = \mathtt{id}_F$ for all i. Notice, however, that it means $\alpha_{i,j} \circ \alpha_{j,i} = \mathtt{id}_{F(j)}$ for all i, j. This implies that $\alpha_{i,j}$ is the inverse of $\alpha_{j,i}$. We have already reached the core of the proof because we now have $F \cong \Delta_{\mathbf{I}}(F(i))$ for all i. What remains is to pay special attention to the trivial case $\mathbf{I} = \varnothing$, and to check that the i-th component of F gives a limit of D. □

Rigid proof. Left as an exercise for the reader. To be written in the near future. □

In conclusion, it is fair to say that the counterexample in [Kel05] is one of the simplest non-pointwise limits.

References

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Reprints in Theory and Applications of Categories*, (10):vi+137, 2005.

[Rie16] Emily Riehl. *Category theory in context*. Aurora Dover Modern Math Originals. Dover Publications, Inc., Mineola, NY, 2016.