Double categories of relations relative to factorization systems and fibrations

Hayato Nasu

Kyoto Univ.

this Sep. $\rightsquigarrow$ Dalhousie Univ.

joint with Keisuke Hoshino (Kyoto Univ.)

CMS Summer Meeting 2025

June 9, 2025

1. Relations and spans in categories

2. Relations relative to factorization systems

3. Relations internal to fibrations

This talk is based on

K. Hoshino, H. Nasu. Double categories of relations relative to factorisation system. (ACS, 2025.)
arXiv: 2310.19428

H. Nasu. Logical Aspects of Virtual double categories, master's thesis, 2025
arXiv: 2501.17869

1. Relations and spans in categories

2. Relations relative to factorization systems

3. Relations internal to fibrations

Double categories

Definition.

A (pseudo) double category consists of

- objects $A, B, \dots$
- vertical arrows $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}, \dots$
- horizontal arrows $A \overset{p}{\rightarrow\!\!\!\!\!\!+} B, \dots$
- cells
$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}, \dots$$

with compositions of

vertical arrows,
$$\begin{tikzcd} A \arrow[d, "f"] \\ B \arrow[d, "g"] \\ C \end{tikzcd}$$

horizontal arrows,
$$\begin{tikzcd} A \arrow[r, "p", "|" marking] & K \arrow[r, "m", "|" marking] & X \end{tikzcd}$$

and cells
$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & K \arrow[r, "m", "|" marking] \arrow[d, "s"] \arrow[dr, phantom, "\nu"] & X \arrow[d, "u"] \\ B \arrow[r, "q"', "|" marking] & L \arrow[r, "n"', "|" marking] & Y \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & K \arrow[d, "s"] \\ B \arrow[r, "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\tau"] & L \arrow[d, "t"] \\ C \arrow[r, "r"', "|" marking] & M \end{tikzcd}$$

subject to some coherence conditions.

Double categories

Example

	$\mathbb{R}\mathrm{el}$	$\mathbb{S}\mathrm{pan}$
Objects	sets	sets
Vertical arrows	functions	functions
Horizontal arrows	relations	spans
Cells $\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle f}\downarrow & \tau & \downarrow{\scriptstyle g} \\ C & \xrightarrow[q]{} & D \end{array}$	A cell exists iff $p(a,b) \Downarrow q(f(a), g(b))$ $(\forall a, b)$	$\begin{array}{ccccc} A & \leftarrow & \lvert p \rvert & \rightarrow & B \\ {\scriptstyle f}\downarrow & \circlearrowleft & \downarrow{\scriptstyle h} & \circlearrowleft & \downarrow{\scriptstyle g} \\ C & \leftarrow & \lvert q \rvert & \rightarrow & D \end{array}$

The horizontal identity and composition in $\mathbb{R}\mathrm{el}$ are :

$$A \xrightarrow{\mathrm{Id}_A} A \qquad\qquad A \xrightarrow{p} B \xrightarrow{q} C$$

$$\mathrm{Id}_A(a, a') :\equiv a = a' \qquad\qquad (p \odot q)(a, c) :\equiv \exists b \in B.\ p(a,b) \wedge q(b,c)$$

Relations and Spans

$\mathcal{C}$: a regular category

$\rightsquigarrow$ $\mathbb{R}\mathrm{el}(\mathcal{C})$: the double category of relations internal to $\mathcal{C}$

Theorem. [Lambert '22]

$\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{C})$ for some $\mathcal{C}$

$\Leftrightarrow$ $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme.

$\mathcal{C}$: a category with fin. limits

$\rightsquigarrow$ $\mathbb{S}\mathrm{pan}(\mathcal{C})$: the double category of spans in $\mathcal{C}$

Theorem. [Aleiferi '19]

$\mathbb{D} \simeq \mathbb{S}\mathrm{pan}(\mathcal{C})$ for some $\mathcal{C}$

$\Leftrightarrow$ $\mathbb{D}$ is a unit-pure cartesian equipment with strong EM-objects for copointed endomorphisms and... .

Can we unify these characterization theorems?

1. Relations and spans in categories

2. Relations relative to factorization systems

3. Relations internal to fibrations

Stable Orthogonal Factorization Systems

Definition.

An **OFS** on $\mathcal{C}$ is a pair (E, M) of $E, M \subseteq \mathrm{mor}(\mathcal{C})$ s.t.

(i) E and M are closed under composition and contain iso's.

(ii) E and M are orthogonal :

$$E \ni \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, "\exists!"] & \bullet \end{tikzcd} \in M$$

(iii) Every morphism in $\mathcal{C}$ is factored as a composite of morphisms in E and M.

An **SOFS** on $\mathcal{C}$ is an OFS with E stable under pullback.

SOFS (E, M) on $\mathcal{C}$ $\rightsquigarrow$ $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$: the double category of M-relations

Relations relative to SOFSs

Definition. $\mathcal{C}$ has fin. limits, (E,M): OFS on $\mathcal{C}$, $I, J \in \mathrm{ob}(\mathcal{C})$

An **M-relation** $I \to\!\!\!\!\!\!+ J$ is a morphism $R \xrightarrow{r} I \times J \in M$.

Proposition. [HN.25] (E,M) : SOFS on $\mathcal{C}$ with fin. limits.

$\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$: the double category of M-relations

$$\begin{tikzcd} I \arrow[r, "{(R,r)}", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & K \arrow[d, "v"] \\ J \arrow[r, "{(S,s)}"', "\shortmid"{marking}] & L \end{tikzcd} \text{ in } \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C}) \quad \leftrightsquigarrow \quad \begin{tikzcd} R \arrow[r, "r"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & I \times K \arrow[d, "u \times v"] \\ S \arrow[r, "s"'] & J \times L \end{tikzcd} \text{ in } \mathcal{C}$$

is a **cartesian equipment**.

Remark. Stability is needed for the associativity of **horizontal composition**.

Characterization theorem

Theorem. [HN. '25]

For a double category $\mathbb{D}$, the following are equivalent.

(i) $\mathbb{D}$ is equivalent to $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ for some finitely complete category $\mathcal{C}$ and an SOFS (E, M) on it.

(ii)
- $\mathbb{D}$ is a cartesian equipment.
- $\mathbb{D}$ has strong tabulators and Beck-Chevalley pullbacks.
- $\mathrm{Fib}(\mathbb{D}) := \left\{ \begin{array}{c} A \\ f \downarrow \\ B \end{array} \;\middle|\; \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "!"] & \\ B \arrow[rr, "\exists R"', "|" marking] & & 1 \end{tikzcd} : \text{a tabulator of } R \right\}$

  is closed under composition.

If these hold, M is "the same" as $\mathrm{Fib}(\mathbb{D})$.

Examples

- $(\mathrm{RegEpi}_{\mathcal{C}}, \mathrm{Mono}_{\mathcal{C}})$ on a regular category $\mathcal{C}$

  $\rightsquigarrow \mathbb{R}\mathrm{el}(\mathcal{C})$ (The characterization thm in [Lambert '22] is recovered from ours.)

- $(\mathrm{Iso}_{\mathcal{C}}, \mathrm{Mor}_{\mathcal{C}})$ on a category with finite limits

  $\rightsquigarrow \mathbb{S}\mathrm{pan}(\mathcal{C})$ (The characterization thm in [Aleiferi '22] derives from ours with additional care.)

- $(\mathrm{Epi}_{\mathcal{C}}, \mathrm{StrMono}_{\mathcal{C}})$ on a quasitopos $\rightsquigarrow$ the DC of strong relations.

1. Relations and spans in categories

2. Relations relative to factorization systems

3. Relations internal to fibrations

Generalizing further

$$
\begin{tikzcd}
\boxed{\text{regular categories } \mathcal{C}} \arrow[d, hook, "{(\mathrm{RegEpi}_{\mathcal{C}},\, \mathrm{Mono}_{\mathcal{C}})}"'] \arrow[rr, bend left, "\text{relations internal to } \mathcal{C}"] & & \\
\boxed{\text{SOFSs } (E, M)} \arrow[d, hook, "{\mathrm{cod}\colon M \to \mathcal{C}}"'] \arrow[rr, "M\text{-relations}"'] & & \boxed{\text{Double categories}} \\
\boxed{\text{fibrations } \mathbb{P} \ (\text{hyperdoctrines})} \arrow[urr, dashed, bend right, "?"'] & &
\end{tikzcd}
$$

relations w.r.t. the internal logic of $\mathbb{P}$ [Pavlovic '96]

Theorem. [Shulman '12]

$\mathbb{P}$: a framable monoidal fibration $\rightsquigarrow$ $\mathbb{F}\mathrm{r}(\mathbb{P})$: a framed bicategory

strongly BC, or weakly BC & internally closed

p-relations

Definition. $p : \mathcal{E} \to \mathcal{B}$: fibration, $\mathcal{B}$ has finite products, $I, J \in \mathcal{B}$

A p-relation $I \nrightarrow J$ is an object $R \in \mathcal{E}_{I \times J}$. $=$ the fiber of $I \times J$

Example.

Q : a preordered set

$\mathrm{Fam}(Q)$: the category of Q-valued sets $\rightsquigarrow$ $\begin{array}{c} \mathrm{Fam}(Q) \\ f_Q \downarrow \\ \mathbf{Set} \end{array}$: fibration

$(a_x)_{x \in X}$ where $a_x \in Q$

f_Q-relations $X \xrightarrow{R} Y$ $=$ Q-valued relations (matrices) $(M_{x,y})_{x \in X, y \in Y}$

Composition of p-relations

Composition of p-relations cannot be defined in general.

Logical understanding : not all fibrations have $\exists$ or $=$ internally.

elementary existential (e.e.) fibrations
$\top, \wedge, =, \exists$

$\cap\mid$

cartesian fibrations (= fibrations with fin. products)
$\top, \wedge$

$\top, \wedge$: fiberwise products

$=$:

$$\begin{tikzcd} \mathcal{E}_I \arrow[r, bend right, "\Sigma_\Delta"'] \arrow[r, phantom, "\top"] & \mathcal{E}_{I\times I} \arrow[l, bend right, "(I \xrightarrow{\Delta} I\times I)^*"'] \end{tikzcd}$$

$\exists$:

$$\begin{tikzcd} \mathcal{E}_{I\times J} \arrow[r, bend right, "\Sigma_\pi"'] \arrow[r, phantom, "\top"] & \mathcal{E}_I \arrow[l, bend right, "(\pi : I\times J \to I)^*"'] \end{tikzcd}$$

Double categories of p-relations

Proposition. [N.] p : e.e. fibration. $\rightsquigarrow$ $\mathbb{B}\mathrm{il}(p)$: a cartesian equipment.

$$\begin{tikzcd} I \arrow[r, "R", "|" marking] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & J \arrow[d, "v"] \\ I' \arrow[r, "R'"', "|" marking] & J' \end{tikzcd} \text{ in } \mathbb{B}\mathrm{il}(p) \quad \Bigg\| \quad \begin{tikzcd} R \arrow[d, "\varphi"'] \\ R' \end{tikzcd} \text{ in } \mathcal{E} \overset{p}{\longmapsto} \begin{tikzcd} I \times J \arrow[d, "u \times v"'] \\ I' \times J' \end{tikzcd} \text{ in } \mathcal{B}$$

$$\begin{tikzcd}[column sep=large] \boxed{\text{SOFSs } (E, M)} \arrow[r, "{\begin{smallmatrix} M \\ \downarrow \mathrm{cod} \\ \mathcal{E} \end{smallmatrix}}"] \arrow[d, phantom, "\cap" rotate=90] & \boxed{\text{e.e. fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d, phantom, "\cap" rotate=90] & \boxed{\text{Cartesian equipments}} \\ \boxed{\text{OFSs on categories with finite limits}} \arrow[r] & \boxed{\text{cartesian fibrations}} \arrow[r] & ? \end{tikzcd}$$

○ p-relations

✕ composition of p-relations

Virtual double categories

Definition. A virtual double category (VDC) $\mathbb{D}$ consists of

- objects $I, J, \ldots$
- vertical arrows $\begin{array}{c} I \\ \downarrow u \\ J \end{array}, \ldots$
- horizontal arrows $I \overset{\alpha}{\nrightarrow} K, \ldots$
- (virtual) cells

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "u"'] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "v"] \\ J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] \arrow[rrr, phantom, "\nu", shift left=4] & & & J_1 \end{tikzcd}$$

for $n \geq 0$

with compositions of vertical arrows and cells that satisfy some laws.

Virtual double categories = Double categories w/o arbitary horizontal composition.

$\rightsquigarrow$ Double categories $\simeq$ VDCs with horizontal composition.

Virtual double categories of $\mathbb{P}$-relations

Observation.

In $\mathbb{R}\mathrm{el}$, a cell of the form
$$\begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "u"'] & Y \arrow[r, "q", "|" marking] \arrow[d, phantom, "\cap\!|"] & Z \arrow[d, "v"] \\ X' \arrow[rr, "r"', "|" marking] & & Z' \end{tikzcd}$$
exists if and only if

$$(\exists y \in Y.\ p(x,y) \wedge q(y,z)) \Rightarrow r(u(x), v(z)) \quad (x \in X,\ z \in Z)$$

but this is equivalent to

$$p(x,y) \wedge q(y,z) \Rightarrow r(u(x), v(z)) \quad (x \in X,\ y \in Y,\ z \in Z).$$

Proposition. [N.]

$\mathbb{P}$: cartesian fibration $\rightsquigarrow$ $\mathbb{B}\mathrm{il}(\mathbb{P})$: a cartesian fibrational VDC

Main result

$$
\begin{tikzcd}
\boxed{\text{e.e. fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d, phantom, sloped, "\subseteq"] \arrow[dr, phantom, very near start, "\lrcorner"] & \boxed{\text{Cartesian equipments}} \arrow[d, phantom, sloped, "\subseteq"] \\
\boxed{\text{cartesian fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \boxed{\text{Cartesian fibrational VDCs}}
\end{tikzcd}
$$

<u>Theorem.</u> [N.] $\quad p$: cartesian fibration

p : an e.e. fibration $\iff$ $\mathbb{B}\mathrm{il}(p)$: a cartesian equipment

"p has $\exists$ and $=$ iff $\mathbb{B}\mathrm{il}(p)$ has composition."

Corollaries

- $\mathcal{C}$: a category with finite limits $\rightsquigarrow$ $\mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$: a cartesian fibration.

  $\mathbb{R}\mathrm{el}(\mathcal{C}) := \mathbb{B}\mathrm{il}(\mathrm{Sub}(\mathcal{C}) \to \mathcal{C})$ is a cartesian equipment

  $\iff$ $\mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$ is an e.e. fibration $\xLeftrightarrow{\text{[Jacobs '99]}}$ $\mathcal{C}$: regular

- Q : a $\wedge$-semilattice $\rightsquigarrow$ $\mathrm{Fam}(Q) \xrightarrow{f_Q} \mathrm{Set}$: a cartesian fibration.

  $Q\text{-}\mathbb{R}\mathrm{el} := \mathbb{B}\mathrm{il}(f_Q)$ is a cartesian equipment

  $\iff$ f_Q is an e.e. fibration $\xLeftrightarrow{\text{[Jacobs '99]}}$ Q : a frame.

Conclusions

$$
\begin{tikzcd}
\boxed{\text{SOFSs } (E,M)} \arrow[r, "{\begin{smallmatrix} M \\ \downarrow \mathrm{cod} \\ E \end{smallmatrix}}"] \arrow[rr, bend left, "\mathbb{R}\mathrm{el}_{(-,-)}(-)"] & \boxed{\text{e.e. fibrations}} \arrow[r] & \boxed{\text{Cartesian equipments}} \\
\boxed{\begin{array}{l}\text{OFSs on categories} \\ \text{with finite limits}\end{array}} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & \boxed{\text{cartesian fibrations}} \arrow[u, phantom, sloped, "\subseteq"] \arrow[u, phantom, shift left=6, "\perp"] \arrow[r, "\mathbb{B}\mathrm{il}"'] & \boxed{\text{Cartesian FVDCs}} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}
$$

Other stuffs & Future work

- Connection to existing results on bicategories & fibrations

  keywords : cartesian bicategories, allegories, hyperdoctrines, exact completion, monoidal fibrations

- Other structures on (virtual) double categories

  keywords : compact/cartesian closures, power objects

Reference

E. Aleiferi. Cartesian Double Categories with an Emphasis on Characterizing Spans. PhD thesis, Dalhousie Univ, 2018.

G.S.H. Crutwell & M. A. Shulman. A unified framework for generalized multicategories. TAC, 24: No.21, 2010.

A. Carboni & R.F.C. Walters. Cartesian bicategories I. JPAA, 49(1-2): 11-32, 1987.

P.J. Freyd & A. Scedrov. Categories, allegories, volume 39 of North-Holland Mathematical Library, 1990.

K. Hoshino & H. Nasu. Double categories of relations relative to factorisation system. (ACS, 2025.)

B. Jacobs. Categorical logic and type theory. volume 141 of SLFM, 1999.

M. Lambert. Double categories of relatios. TAC. 38, No.33. 2022.

F. Lawler. Fibrations of Predicates and Bicategories of Relations, Phd thesis. Trinity College 2015

D. Pavlović. Maps II. Chasing diagrams in categorical proof theory. J. IGPL, 4(2): 159-194, 1996

M. Shulman. Framed bicategories and monoidal fibrations TAC. 20, No.18. 2008.

Thank you!

https://hayatonasu.github.io/hayatonasu/

Characterizing $\mathbb{B}\mathrm{il}(\mathbb{P})$

A1 / A2

$$
\begin{tikzcd}
\boxed{\text{SOFSs } (E,M)} \arrow[r, "{\substack{M\\ \downarrow \mathrm{cod}\\ E}}"] \arrow[rr, bend left, "{\mathbb{R}\mathrm{el}_{(-,-)}(-)}"] \arrow[d, phantom, sloped, "\subseteq"] & \boxed{\text{e.e. fibrations}} \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{\text{Cartesian equipments}} \arrow[d, phantom, sloped, "\subseteq"] \\
\boxed{\text{OFSs on categories with finite limits}} \arrow[r] & \boxed{\text{cartesian fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \boxed{\text{Cartesian FVDCs}}
\end{tikzcd}
$$

For the characterization result, we have :

Theorem. [N.]

$$\mathcal{E}\mathcal{E}\mathrm{Fib} \xrightarrow[\simeq]{\mathbb{B}\mathrm{il}} \mathrm{Cart}\mathcal{E}\mathrm{q}_{\mathrm{Frob}} \subseteq \mathrm{Cart}\mathcal{E}\mathrm{q}$$

the 2-cat. of Frobenius cartesian equipments.

This will be the topic of my talk at FMCS 2025 !

Definitions

A fibration p is **cartesian** if
- B has *finite products*, and
- all fibers $\mathcal{E}_I$ have *finite products* preserved by the base change functors.

p is **elementary existential (e.e.)** if
- it is cartesian,
- the base change functors along $I \times J \xrightarrow{\pi} I$ & $I \times J \xrightarrow{id \times \Delta} I \times J \times J$'s have *left adjoints*, and
- BC condition and Frobenius reciprocity hold for these adjoints.

A **restriction** of
$$\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "\gamma"', "\shortmid" marking] & D \end{tikzcd}$$
is the universal cell
$$\begin{tikzcd} A \arrow[r, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\bullet"] & C \arrow[d, "g"] \\ B \arrow[r, "\gamma"', "\shortmid" marking] & D \end{tikzcd}.$$

A VDC is **fibrational** if it admits all restrictions.

A fibrational VDC is **cartesian** if the right adjoints below exist.

$$\mathbb{D} \underset{\perp}{\overset{!}{\rightleftarrows}} \mathbb{1}, \quad \mathbb{D} \underset{\perp}{\overset{\Delta}{\rightleftarrows}} \mathbb{D} \times \mathbb{D} \quad \text{in } \mathcal{FVDC}.$$