

# Double categories of relations relative to factorization systems and fibrations

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this Sep.

~~~~~→ Dalhousie Univ.

joint with Keisuke Hoshino (Kyoto Univ.)

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1. Relations and spans in categories
2. Relations relative to factorization systems
3. Relations internal to fibrations

This talk is based on

K. Hoshino, H. Nasu. Double categories of relations relative to factorisation system. (ACS, 2025.)

arXiv: 2310.19428

H. Nasu. Logical Aspects of Virtual double categories, master's thesis, 2025

arXiv: 2501.17869

1. Relations and spans in categories
2. Relations relative to factorization systems
3. Relations internal to fibrations

# Double categories

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## Definition.

A (pseudo) double category consists of

- objects  $A, B, \dots$

- vertical arrows  $A \downarrow f B, \dots$

- horizontal arrows  $A \xrightarrow{p} B, \dots$

- cells  $\begin{array}{ccc} A & \xrightarrow{p} & B \\ f \downarrow & \sigma & \downarrow g \\ C & \xrightarrow{q} & D \end{array}, \dots$

with compositions of vertical arrows,

$$\begin{array}{c} A \\ \downarrow f \\ B \\ \downarrow g \\ C \end{array}$$

horizontal arrows,

$$A \xrightarrow{p} K \xrightarrow{m} X$$

and cells

$$\begin{array}{ccccc} A & \xrightarrow{p} & K & \xrightarrow{m} & X \\ f \downarrow & \sigma & \downarrow s & \nu & \downarrow u \\ B & \xrightarrow{q} & L & \xrightarrow{n} & Y \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{p} & K & & \\ f \downarrow & \sigma & \downarrow s & & \\ B & \xrightarrow{\quad} & L & & \\ g \downarrow & \tau & \downarrow t & & \\ C & \xrightarrow{r} & M & & \end{array}$$

subject to some coherence conditions.

# Double categories

## Example

|                   | Rel                                                                                                                                                                                                                                                                        | Span      |
|-------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|
| Objects           | sets                                                                                                                                                                                                                                                                       | sets      |
| Vertical arrows   | functions                                                                                                                                                                                                                                                                  | functions |
| Horizontal arrows | relations                                                                                                                                                                                                                                                                  | spans     |
| Cells             | <p>A cell exists iff</p> $  \begin{array}{ccc}  A & \xrightarrow{p} & B \\  f \downarrow & \tau & \downarrow g \\  C & \xrightarrow{q} & D  \end{array}  $ <p> <math>p(a, b) \quad (\forall a, b)</math><br/> <math>\Downarrow</math><br/> <math>q(f(a), g(b))</math> </p> |           |
|                   | $  \begin{array}{ccccc}  A & \leftarrow  p  & & & \rightarrow B \\  f \downarrow & \wr & \downarrow h & \wr & \downarrow g \\  C & \leftarrow  q  & & & \rightarrow D  \end{array}  $                                                                                      |           |

The horizontal identity and composition in Rel are :

$$A \xrightarrow{\text{Id}_A} A$$

$$\text{Id}_A(a, a') \equiv a = a'$$

$$A \xrightarrow{p} B \xrightarrow{q} C$$

$$(p \odot q)(a, c) \equiv \exists b \in B. p(a, b) \wedge q(b, c)$$

# Relations and Spans

$\mathcal{C}$  : a regular category



$\mathbf{Rel}(\mathcal{C})$  : the double category of  
relations internal to  $\mathcal{C}$

Theorem. [Lambert '22]

$\mathbb{D} \simeq \mathbf{Rel}(\mathcal{C})$  for some  $\mathcal{C}$

$\mathbb{D}$  is a 'double category of  
 $\Leftrightarrow$  relations' with a subobject comprehension scheme.

$\mathcal{C}$  : a category with fin. limits



$\mathbf{Span}(\mathcal{C})$  : the double category of  
spans in  $\mathcal{C}$

Theorem. [Aleiferi '19]

$\mathbb{D} \simeq \mathbf{Span}(\mathcal{C})$  for some  $\mathcal{C}$

$\mathbb{D}$  is a unit-pure cartesian  
 $\Leftrightarrow$  equipment with strong EM-objects  
for copointed endomorphisms and...

Can we unify these characterization theorems?

1. Relations and spans in categories
2. Relations relative to factorization systems
3. Relations internal to fibrations

# Stable Orthogonal Factorization Systems

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## Definition.

An **OFS** on  $\mathcal{C}$  is a pair  $(E, M)$  of  $E, M \subseteq \text{mor}(\mathcal{C})$  s.t.

(i)  $E$  and  $M$  are closed under composition and contain iso's.

(ii)  $E$  and  $M$  are orthogonal :

$$E \ni \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \text{\textcolor{red}{\exists!}} & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \in M$$

(iii) Every morphism in  $\mathcal{C}$  is factored as a composite of morphisms in  $E$  and  $M$ .

An **SOFS** on  $\mathcal{C}$  is an OFS with  $E$  stable under pullback.

SOFS  
 $(E, M)$  on  $\mathcal{C} \rightsquigarrow \text{Rel}_{(E, M)}(\mathcal{C})$  : the double category of  
M-relations



# Relations relative to SOFSs

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Definition.  $\mathcal{C}$  has fin.limits,  $(E, M)$ : OFS on  $\mathcal{C}$ ,  $I, J \in \text{ob}(\mathcal{C})$

An **M-relation**  $I \rightharpoonup J$  is a morphism  $R \xrightarrow{r} I \times J \in M$ .

Proposition. [HN.25]  $(E, M)$ : SOFS on  $\mathcal{C}$  with fin.limits.

$\text{Rel}_{(E, M)}(\mathcal{C})$ : the double category of M-relations

$$\begin{array}{ccc} I & \xrightarrow{(R, r)} & K \\ u \downarrow & \alpha & \downarrow v \\ J & \xrightarrow{(S, s)} & L \end{array} \text{ in } \text{Rel}_{(E, M)}(\mathcal{C}) \quad \longleftrightarrow \quad \begin{array}{ccc} R & \xrightarrow{r} & I \times K \\ \alpha \downarrow & \alpha & \downarrow u \times v \\ S & \xrightarrow{s} & J \times L \end{array} \text{ in } \mathcal{C}$$

is a **cartesian equipment**.

Remark. Stability is needed for the associativity of **horizontal composition**.

# Characterization theorem

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## Theorem. [HN. 25]

For a double category  $\mathbb{D}$ , the following are equivalent.

(i)  $\mathbb{D}$  is equivalent to  $\mathbf{Rel}_{(E,M)}(\mathcal{C})$

for some finitely complete category  $\mathcal{C}$  and an SOFS  $(E, M)$  on it.

(ii) •  $\mathbb{D}$  is a **cartesian equipment**.

•  $\mathbb{D}$  has strong tabulators and Beck-Chevalley pullbacks.

•  $\text{Fib}(\mathbb{D}) := \left\{ \begin{array}{c} A \\ f \downarrow \\ B \end{array} \mid \begin{array}{ccc} & A & \\ f \swarrow & & \searrow ! \\ B & \xrightarrow[R]{\exists} & 1 \end{array} : \text{a tabulator of } R \right\}$

is closed under composition.

If these hold,  $M$  is "the same" as  $\text{Fib}(\mathbb{D})$ .

# Examples

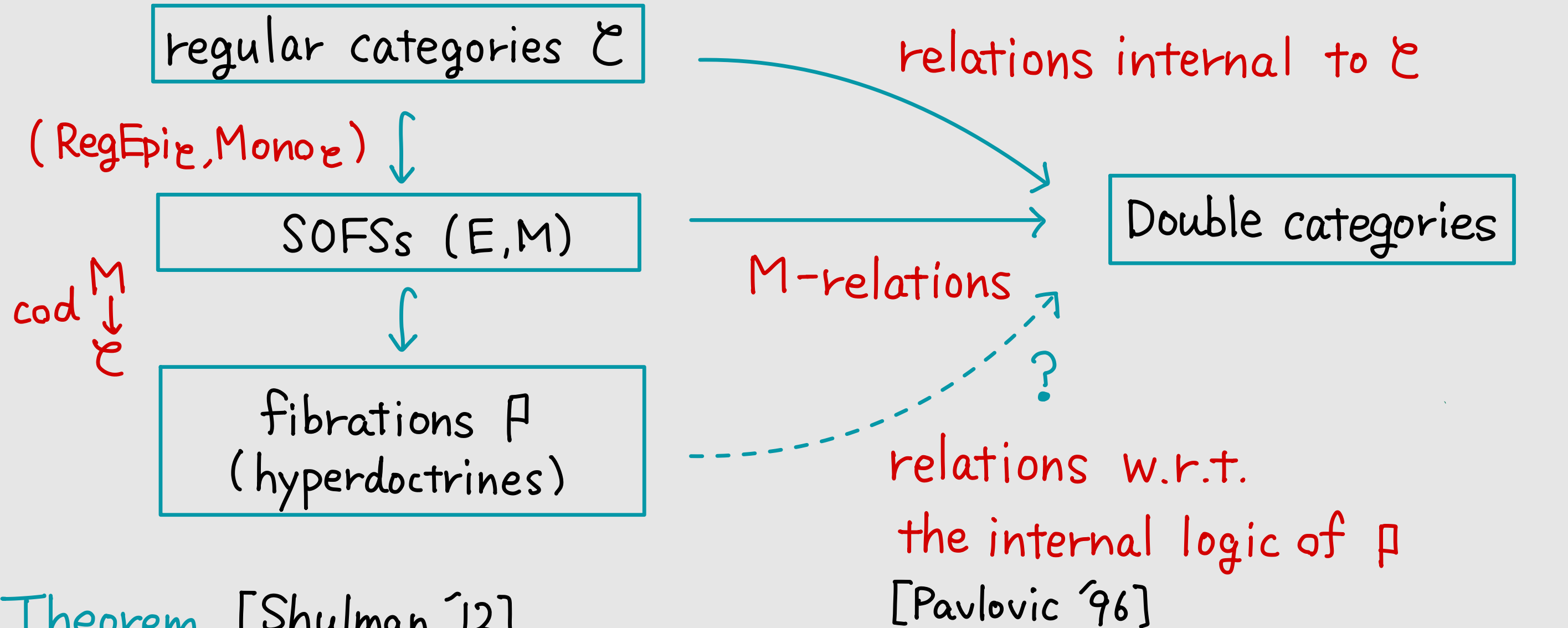
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- $(\text{RegEpi}_{\mathcal{C}}, \text{Mono}_{\mathcal{C}})$  on a regular category  $\mathcal{C}$   
 $\rightsquigarrow \text{Rel}(\mathcal{C})$   $\left( \begin{array}{l} \text{The characterization thm in [Lambert '22]} \\ \text{is recovered from ours.} \end{array} \right)$
- $(\text{Iso}_{\mathcal{C}}, \text{Mono}_{\mathcal{C}})$  on a category with finite limits  
 $\rightsquigarrow \text{Span}(\mathcal{C})$   $\left( \begin{array}{l} \text{The characterization thm in [Aleiferi '22]} \\ \text{derives from ours with additional care.} \end{array} \right)$
- $(\text{Epi}_{\mathcal{C}}, \text{StrMono}_{\mathcal{C}})$  on a quasitopos  $\rightsquigarrow$  the DC of **strong relations**.

1. Relations and spans in categories
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# Generalizing further

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Theorem. [Shulman '12]

$\mathbb{P}$  : a framable monoidal fibration  $\rightsquigarrow \text{Fr}(\mathbb{P})$  : a framed  
Strongly BC, or weakly BC & internally closed bicategory

# $\mathcal{P}$ -relations

Definition.  $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{B}$  : fibration,  $\mathcal{B}$  has finite products,  $I, J \in \mathcal{B}$

A  $\mathcal{P}$ -relation  $I \rightarrowtail J$  is an object  $R \in \mathcal{E}_{I \times J}$   $\equiv$  the fiber of  $I \times J$

Example.

$\mathcal{Q}$  : a preordered set

$\text{Fam}(\mathcal{Q})$  : the category of  $\mathcal{Q}$ -valued sets  $\leadsto \begin{array}{c} \text{Fam}(\mathcal{Q}) \\ \downarrow \text{I}_{\mathcal{Q}} \\ \text{Set} \end{array}$  : fibration  
 $(a_x)_{x \in X}$  where  $a_x \in \mathcal{Q}$

$\text{I}_{\mathcal{Q}}$ -relations  $X \xrightarrow{R} Y = \mathcal{Q}$ -valued relations  $(M_{x,y})_{x \in X, y \in Y}$   
 (matrices)

# Composition of $\beta$ -relations

Composition of  $\beta$ -relations cannot be defined in general.

**Logical understanding** : not all fibrations have  $\exists$  or  $=$  internally.

elementary existential  
(e.e.) fibrations  
 $\top, \wedge, =, \exists$

$\cup$

cartesian fibrations  
(= fibrations with fin. products)  
 $\top, \wedge$

$\top, \wedge$  : fiberwise products

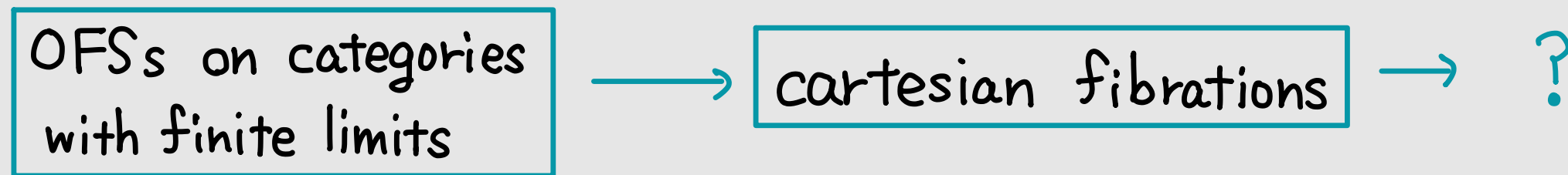
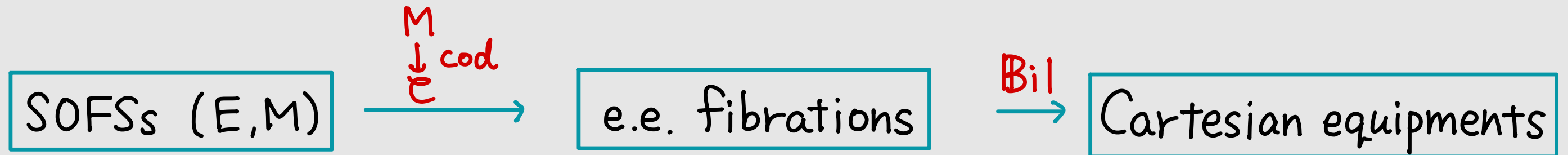
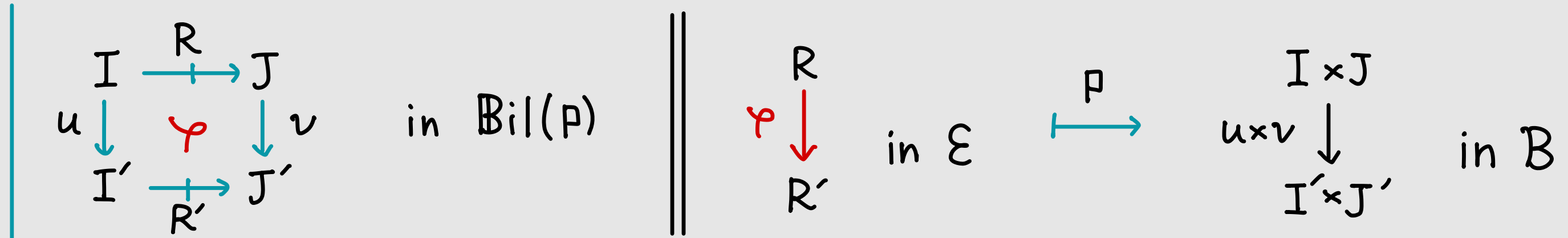
$$= : \quad \begin{array}{ccc} & (I \xrightarrow{\Delta} I \times I)^* & \\ \mathcal{E}_I & \begin{array}{c} \xleftarrow{\quad \top \quad} \\ \xrightarrow{\quad \Sigma_{\Delta} \quad} \end{array} & \mathcal{E}_{I \times I} \end{array}$$

$$\exists : \quad \begin{array}{ccc} & (\pi : I \times J \rightarrow I)^* & \\ \mathcal{E}_{I \times J} & \begin{array}{c} \xleftarrow{\quad \top \quad} \\ \xrightarrow{\quad \Sigma_{\pi} \quad} \end{array} & \mathcal{E}_I \end{array}$$

# Double categories of $\beta$ -relations

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Proposition. [N.]  $\beta : \text{e.e.fibration.} \rightsquigarrow \text{Bil}(\beta) : \text{a cartesian equipment.}$



$\circ$   $\beta$ -relations

$\times$  composition of  $\beta$ -relations



# Virtual double categories

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Definition. A **virtual double category (VDC)**  $\mathbb{D}$  consists of

- objects  $I, J, \dots$
- vertical arrows  $\begin{matrix} I \\ \downarrow u \\ J \end{matrix}, \dots$
- horizontal arrows  $I \xrightarrow{\alpha} K, \dots$

• **(virtual) cells**

$$\begin{array}{ccccccc} & & \alpha_1 & & & \alpha_n & \\ & & \xrightarrow{\quad} & & \xrightarrow{\quad} & \xrightarrow{\quad} & \\ I_0 & \xrightarrow{\quad} & I_1 & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & I_n \\ \downarrow u & & & \downarrow \nu & & & \downarrow \nu \\ J_0 & \xrightarrow{\quad} & & & \xrightarrow{\quad} & & J_1 \\ & & \beta & & & & \end{array} \quad \text{for } n \geq 0$$

with compositions of vertical arrows and cells that satisfy some laws.

Virtual double categories = Double categories  
w/o arbitrary horizontal composition.

$\rightsquigarrow$  Double categories  $\simeq$  VDCs with horizontal composition.

## Observation.

In  $\mathbb{R}el$ , a cell of the form

$$\begin{array}{ccccc} X & \xrightarrow{p} & Y & \xrightarrow{q} & Z \\ u \downarrow & & \text{nl} & & \downarrow v \\ X' & \xrightarrow{r} & & & Z' \end{array}$$

exists if and only if

$$(\exists y \in Y. p(x, y) \wedge q(y, z)) \Rightarrow r(u(x), v(z)) \quad (x \in X, z \in Z)$$

but this is equivalent to

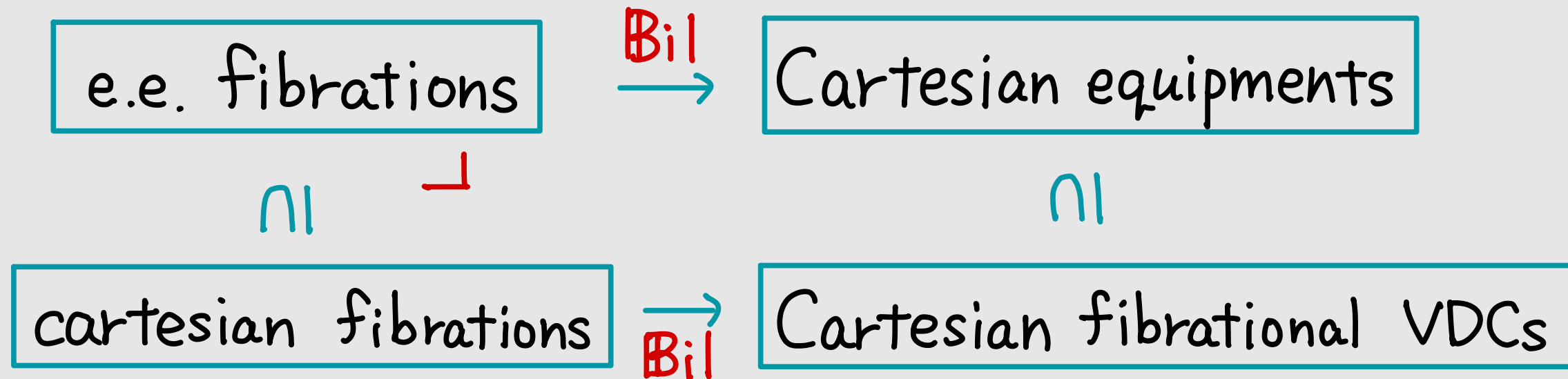
$$p(x, y) \wedge q(y, z) \Rightarrow r(u(x), v(z)) \quad (x \in X, y \in Y, z \in Z).$$

## Proposition. [N.]

$\mathbb{P}$ : cartesian fibration  $\rightsquigarrow \mathbb{B}il(\mathbb{P})$ : a cartesian fibrational VDC

# Main result

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Theorem. [N.]  $\mathcal{P}$  : cartesian fibration

$\mathcal{P}$  : an e.e. fibration  $\iff \text{Bil}(\mathcal{P})$  : a cartesian equipment

“ $\mathcal{P}$  has  $\exists$  and  $=$  iff  $\text{Bil}(\mathcal{P})$  has composition.”

# Corollaries

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- $\mathcal{C}$  : a category with finite limits  $\rightsquigarrow \text{Sub}(\mathcal{C}) \rightarrow \mathcal{C}$  : a cartesian fibration.

$\text{Rel}(\mathcal{C}) := \text{Bil}(\text{Sub}(\mathcal{C}) \rightarrow \mathcal{C})$  is a cartesian equipment

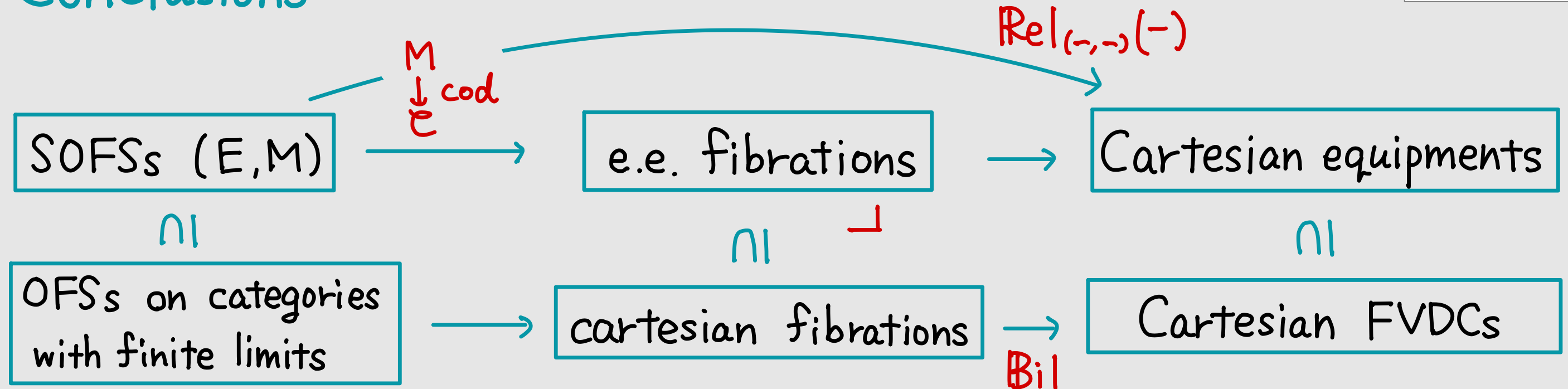
$\Leftrightarrow \text{Sub}(\mathcal{C}) \rightarrow \mathcal{C}$  is an e.e. fibration  $\xLeftrightarrow[\text{Jacobs '99}] \mathcal{C} : \text{regular}$

- $Q$  : a  $\wedge$ -semilattice  $\rightsquigarrow \text{Fam}(Q) \xrightarrow{\text{f}_Q} \text{Set}$  : a cartesian fibration.

$Q\text{-Rel} := \text{Bil}(\text{f}_Q)$  is a cartesian equipment

$\Leftrightarrow \text{f}_Q$  is an e.e. fibration  $\xLeftrightarrow[\text{Jacobs '99}] Q : \text{a frame.}$

# Conclusions



## Other stuffs & Future work

- Connection to existing results on bicategories & fibrations  
 keywords: cartesian bicategories, allegories, hyperdoctrines, exact completion, monoidal fibrations
- Other structures on (virtual) double categories  
 keywords: compact/cartesian closures, power objects

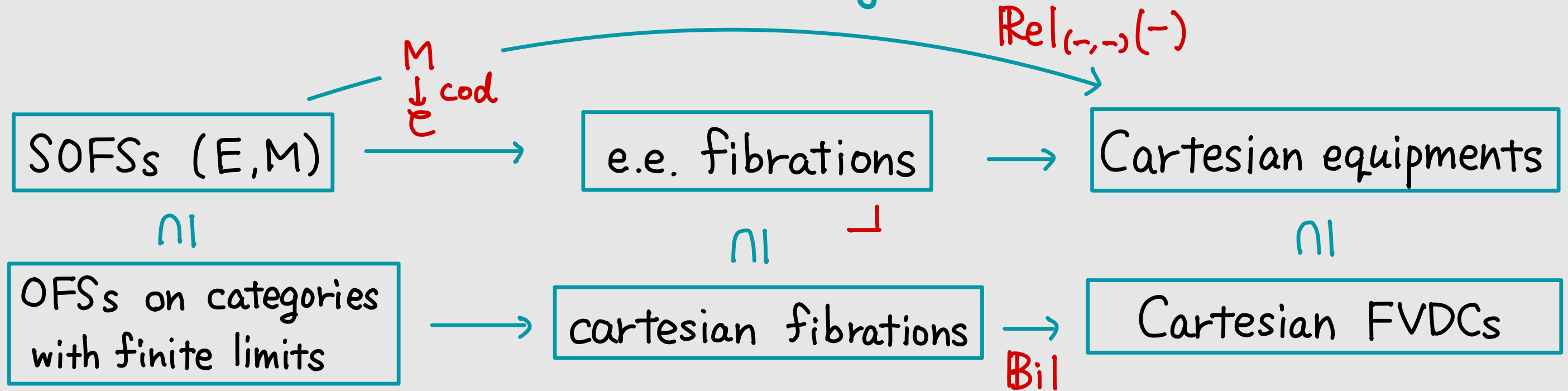
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Thank you!

<https://hayatonasu.github.io/hayatonasu/>

# Characterizing $\text{Bil}(\mathbb{P})$



For the characterization result, we have :

## Theorem. [N.]

$$\mathcal{E} \mathcal{E} \text{Fib} \xrightarrow[\cong]{\text{Bil}} \text{Cart} \mathcal{E} q_{\text{Frob}} \subseteq \text{Cart} \mathcal{E} q$$

the 2-cat. of Frobenius cartesian equipments.

This will be the topic of my talk at FMCS 2025!



# Definitions

A2 / A2

A fibration  $p$  is **cartesian** if

- $B$  has **finite products**, and
- all fibers  $E_I$  have **finite products** preserved by the base change functors.

$p$  is **elementary existential (e.e.)** if

- it is cartesian,
- the base change functors along  $I \times J \xrightarrow{\pi} I$  &  $I \times J \xrightarrow{id \times \Delta} I \times J \times J$  have **left adjoints**, and
- BC condition and Frobenius reciprocity hold for these adjoints.

A **restriction** of  $f \downarrow B \xrightarrow[\gamma]{+} D$  is

the universal cell  $f \downarrow B \xrightarrow[\gamma]{+} D$  with a universal cell  $A \xrightarrow{+} C$  and  $C \downarrow D$ .

A VDC is **fibrational** if it admits all restrictions.

A fibrational VDC is **cartesian** if the right adjoints below exist.

$$D \xleftarrow[\perp]{!} 1, \quad D \xleftarrow[\perp]{\Delta} D \times D \quad \text{in } FVDC.$$