Cartesian Bicategories and Cartesian Equipments

Categorical Logic Meets Double Categories - Side B

Hayato Nasu
Kyoto University
hnasu@kurims.kyoto-u.ac.jp
hayatonasu.github.io

CSCAT 2025 @崇城大学
Mar. 10th 2025

This talk is based on my recent thesis:

Logical Aspects of Virtual Double Categories

Chapter 2: Categorical Logic meets Double Categories

Slides

Outline

1. Cartesian bicategories
2. Cartesian equipments
3. Double categories of relations

"Having finite products".

Outline

1. Cartesian bicategories
2. Cartesian equipments
3. Double categories of relations

Target examples

Bicategory	Rel	Span	Prof
objects (0-cells)	sets $A, B, \ldots$	sets $A, B, \ldots$	categories $A, B, \ldots$
1-cells $A \nrightarrow B$	relations $R \subseteq A \times B$	spans $A \xleftarrow{f} X \xrightarrow{g} B$	profunctors $P : A^{op} \times B \to \mathrm{Set}$
2-cells $A \Downarrow B$	inclusion $R \subseteq S \; (\subseteq A \times B)$	span maps $(A \xleftarrow{f} X \xrightarrow{g} B) \overset{t}{\Downarrow} (A \xleftarrow{h} Y \xrightarrow{k} B)$	transformations $A^{op} \times B \underset{Q}{\overset{P}{\Downarrow \tau}} \mathrm{Set}$
products (?) & local products	$A \times B$ $R \cap S$	$A \times B$ fiber product $X \times_{A \times B} Y$	$A \times B$ $P \times Q$

Cartesian bicategories

Cartesian bicategories were introduced as a framework to capture the "products" in question and unify **Rel**, **Span**, **Prof**, ... [CW'87].

PROBLEM

The "products" of 0-cells in the previous slide are NOT the products in the bicategorical sense.

Example In **Rel**, $A+B$ is the product of A and B.

$$C \xrightarrow[\mid]{R} A+B \quad \Bigg\| \quad \begin{pmatrix} C \xrightarrow[\mid]{R_1} A \\ C \xrightarrow[\mid]{R_2} B \end{pmatrix}$$

Then, what are those "products"?

In $\mathbf{Rel}$, $A \times B$ is the finite product in the category of functions.

What are they in terms of $\mathbf{Rel}$?

$\boxed{\text{Fact}}$ $A \overset{R}{\nrightarrow} B$ in $\mathbf{Rel}$ is a functional relation

$\iff$ R is a left adjoint in $\mathbf{Rel}$

$\boxed{\text{Def}}$ A 1-cell in a bicategory is called a map if it allows a right adjoint.

$\mathbf{Map}(\mathbf{B}) \subseteq \mathbf{B}$ is the wide locally-full subbicategory spanned by maps.

$A \times B$ seems the finite product in $\mathbf{Map}(\mathbf{B})$.

Definitions of cartesian bicategories

'87 Carboni, Walters "Cartesian bicategories I"

(limited to locally posetal bicategories.)

'08 Carboni, Kelly, Walters, Wood "Cartesian bicategories II"

'09 Todd Trimble's post on nLab.

Def ① [CW'87]
A cartesian structure on a locally posetal bicategory $\mathcal{B}$ consists of
- a monoidal product $\otimes$ on $\mathcal{B}$
- a unique cocommutative comonoid structure $(\Delta_X : X \to X \otimes X,\ t_X : X \to I)$ for which Δ_X, t_X have right adjoints on every $X \in \mathcal{B}$

s.t. every 1-cell is a lax comonoid homomorphism.

Def (CKWW '08)
A bicategory $\mathcal{B}$ is precartesian if
(i) the hom-categories $\mathcal{B}(A, B)$ have finite products for all A, B.
(ii) the bicategory $\mathbf{Map}(\mathcal{B})$ has finite products in the sense of bilimit.

Prop [CKWW '08]
For a precartesian bicategory $\mathcal{B}$, lax functors $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $I : 1 \to \mathcal{B}$ are induced.

Def ② A precartesian bicategory is cartesian if $\otimes$ and I above are pseudo functors.

Def ③ A cartesian strcture on $\mathcal{B}$ consists of
- $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $I : 1 \to \mathcal{B}$
- transformations valued on maps $\left(\varepsilon_A : A \to I,\ \delta_A : A \to A \otimes A,\ \pi_{A,B} : A \otimes B \to A,\ \pi'_{A,B} : A \otimes B \to B \right)$
- invertible modification with some coherent conditions.

Why is it so hard?

Problem: the universal properties of the "products" are for maps, not all 1-cells

|| Left adjoints in the bicategory

Remember...

a category $\mathcal{C}$ is cartesian

(= has finite products)

if $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{C}\times\mathcal{C} \arrow[l, bend left, "\exists\times"] \end{tikzcd}$ & $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\exists 1"] \end{tikzcd}$.

- Easy to check.
- Property-like structure.
- Only in terms of the 2-cat. $\mathcal{CAT}$.

Def (cartesian objects, CKW'91)

$\mathcal{K}$: 2-category with finite products

$C \in \mathcal{K}$ is a cartesian object

if $\begin{tikzcd} C \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & C\times C \arrow[l, bend left, "\exists\times"] \end{tikzcd}$ & $\begin{tikzcd} C \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\exists 1"] \end{tikzcd}$

It is unknow whether cartesian bicategories can be formulated as cartesian objects in any 2-category.

Questioning bicategories

Maps themselves are not the primitive data of bicategories, making it difficult to formulate universal properties for them.

Then, why don't we separate "maps" from left adjoints as a new kind of 1-cells, and define the finite products in terms of these 1-cells?

$\Downarrow$

Double categories!!

Outline

1. Cartesian bicategories
2. Cartesian equipments
3. Double categories of relations

Double categories

Def

A (pseudo) double category consists of

- objects $A, B, \dots$
- vertical arrows $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$, ...
- horizontal arrows $A \xrightarrow{p} B$, ...
- cells $\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$, ...

with compositions of

vertical arrows, $\begin{tikzcd} A \arrow[d, "f"] \\ B \arrow[d, "g"] \\ C \end{tikzcd}$

horizontal arrows,

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] & K \arrow[r, "m", "\shortmid"{marking}] & X \end{tikzcd}$$

and cells

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & K \arrow[r, "m", "\shortmid"{marking}] \arrow[d, "s"] \arrow[dr, phantom, "\nu"] & X \arrow[d, "u"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & L \arrow[r, "n"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & K \arrow[d, "s"] \\ B \arrow[r, "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\tau"] & L \arrow[d, "t"] \\ C \arrow[r, "r"', "\shortmid"{marking}] & M \end{tikzcd}$$

subject to some coherent conditions.

Double categories

Example

Example	$\mathbb{R}\mathrm{el}$	$\mathbb{S}\mathrm{pan}$	$\mathbb{P}\mathrm{rof}$
Objects	sets	sets	categories
Vertical arrows	functions	functions	functors
Horizontal arrows	relations	spans	profunctors

Cells

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\tau"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

$\mathbb{R}\mathrm{el}$: A cell exists iff

$$p(a,b) \Downarrow q(f(a), g(b)) \quad (\forall a,b)$$

$\mathbb{S}\mathrm{pan}$:

$$\begin{tikzcd} A \arrow[d, "f"'] & {|p|} \arrow[l] \arrow[r] \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "g"] \\ C & {|q|} \arrow[l] \arrow[r] & D \end{tikzcd}$$

$\mathbb{P}\mathrm{rof}$:

$$\begin{tikzcd} A^{op}\times B \arrow[dr, "p"] \arrow[dd, "f^{op}\times g"'] \arrow[dd, phantom, shift left=4, "\Downarrow\tau"] & \\ & \mathrm{Set} \\ C^{op}\times D \arrow[ur, "q"'] & \end{tikzcd}$$

Remark

The horizontal composition is required to be associative only up to isomorphism.

e.g. the composition of spans

$$\begin{tikzcd} & & {|p \circ q|} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & {|p|} \arrow[dl] \arrow[dr] & & {|q|} \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

Notation

$$
\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\tau"] & \\
B \arrow[rr, "p"', "|" marking] & & C
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A \arrow[r, equal, "\mathrm{Id}_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\tau"] & A \arrow[d, "g"] \\
B \arrow[r, "p"', "|" marking] & C
\end{tikzcd}
$$

(e.g. In $\mathbb{R}\mathrm{el}$, the cell implies that

$$
\frac{a = a' \Longrightarrow p(f(a), g(a')) \quad (\forall a, a')}{\Longrightarrow p(f(a), g(a)) \quad (\forall a)}
$$
)

Similar for
$$
\begin{tikzcd}
A \arrow[rr, "p", "|" marking] \arrow[dr, "f"'] & \arrow[d, phantom, "\tau"] & B \arrow[dl, "g"] \\
 & C &
\end{tikzcd}
$$
.

This convention is justified by AVDCs.
(See the next talk by Yuto Kawase.)

Equipments

In $\mathbb{R}\mathrm{el}$, a function f gives rise to a functional relation R_f.

Cells describe this situation as

$$
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\cap\vert"] & \\
A \arrow[rr, "R_f"', "|" marking] & & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[rr, "R_f", "|" marking] \arrow[dr, "f"'] & \arrow[d, phantom, "\cap\vert"] & B \arrow[dl, equal] \\
 & B &
\end{tikzcd}
$$

$R_f(a, f(a))$ $\qquad$ $R_f(a,b) \Rightarrow f(a) = b$

$$
+ \quad
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\cap\vert"] & \\
A \arrow[rr, "R_f"', "|" marking] \arrow[dr, "f"'] & \arrow[d, phantom, "\cap\vert"] & B \arrow[dl, equal] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "="] \\
B
\end{tikzcd}
,\quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\cap\vert"] & A \arrow[r, "R_f", "|" marking] \arrow[dr, "f"] \arrow[d, phantom, "\cap\vert"] & B \arrow[d, equal] \\
A \arrow[r, "R_f"', "|" marking] & B \arrow[r, equal] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "R_f", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[r, "R_f"', "|" marking] & B
\end{tikzcd}
$$

Equipments

$\boxed{\text{Def}}$ [Shulman '08, Aleiferi '19]

A double category is an *equipment* if every vertical arrow $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$ admits

$$\begin{tikzcd} A \arrow[r, "f_*", "\shortmid"{marking}] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "f^*", "\shortmid"{marking}] & A \end{tikzcd} \quad \text{with}$$

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[rr, "f_*"', "\shortmid"{marking}] & & B \end{tikzcd}, \quad \begin{tikzcd} A \arrow[rr, "f_*", "\shortmid"{marking}] \arrow[dr, "f"'] & \arrow[d, phantom, "\beta"] & B \arrow[dl, equal] \\ & B & \end{tikzcd}, \quad \text{s.t.}$$

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, equal] \arrow[d, phantom, "\gamma"] & \\ B \arrow[rr, "f^*"', "\shortmid"{marking}] & & A \end{tikzcd}, \quad \begin{tikzcd} B \arrow[rr, "f^*", "\shortmid"{marking}] \arrow[dr, equal] & \arrow[d, phantom, "\delta"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[rr, "\shortmid"{marking}] \arrow[dr, "f"'] & \arrow[d, phantom, "\beta"] & B \arrow[dl, equal] \\ & B & \end{tikzcd} = \begin{tikzcd} A \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "="] \\ B \end{tikzcd} \quad \&$$

$$\begin{tikzcd} A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, "f"] \arrow[dr, phantom, shift right=3ex, "\alpha"] \arrow[dr, phantom, shift left=3ex, "\beta"] & B \arrow[d, equal] \\ A \arrow[r, "f_*"', "\shortmid"{marking}] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\ A \arrow[r, "f_*"', "\shortmid"{marking}] & B \end{tikzcd} \quad \text{and their duals hold.}$$

Example

In $\mathbb{P}\mathrm{rof}$, for $F : \mathcal{C} \longrightarrow \mathcal{D}$

$$F^*(\bullet, \circ) := \mathcal{D}(F\bullet, \circ) : \mathcal{C} \mathrel{\relbar\mkern-2mu\mapstochar\mkern-2mu\rightarrow} \mathcal{D}$$

$$F_*(\circ, \bullet) := \mathcal{D}(\circ, F\bullet) : \mathcal{D} \mathrel{\relbar\mkern-2mu\mapstochar\mkern-2mu\rightarrow} \mathcal{C}$$

Restrictions in equipments

Example

For $\begin{tikzcd} A' & B' \\ A \arrow[from=1-1, "f"'] \arrow[r, "R"', "\shortmid" marking] & B \arrow[from=1-2, "g"] \end{tikzcd}$ in $\mathbb{R}\mathrm{el}$,

there is the largest relation $A' \overset{S}{\nrightarrow} B'$

s.t. $\begin{tikzcd} A' \arrow[r, "S", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\cap\!\shortmid"] & B' \arrow[d, "g"] \\ A \arrow[r, "R"', "\shortmid" marking] & B \end{tikzcd}$

Namely,

$$R[f;g](a,b) \overset{\text{def}}{\iff} R(f(a), g(b)).$$

This is achevied with the composite

$$A' \overset{f_*}{\nrightarrow} A \overset{R}{\nrightarrow} B \overset{g^*}{\nrightarrow} B'.$$

Def

A restriction of $\begin{tikzcd} A' & B' \\ A \arrow[from=1-1, "f"'] \arrow[r, "p"', "\shortmid" marking] & B' \arrow[from=1-2, "g"] \end{tikzcd}$ is

$A' \overset{p[f;g]}{\nrightarrow} B'$ together with $\begin{tikzcd} A' \arrow[r, "{p[f;g]}", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & B' \arrow[d, "g"] \\ A \arrow[r, "p"', "\shortmid" marking] & B' \end{tikzcd}$

that shows the following universal property:

$$\begin{tikzcd} X \arrow[r, "q", "\shortmid" marking] \arrow[d, "x"'] \arrow[ddr, phantom, "\xi"] & Y \arrow[d, "y"] \\ A' \arrow[d, "f"'] & B' \arrow[d, "g"] \\ A \arrow[r, "p"', "\shortmid" marking] & B \end{tikzcd} = \exists! \begin{tikzcd} X \arrow[r, "q", "\shortmid" marking] \arrow[d, "x"'] \arrow[dr, phantom, "\tilde{\xi}"] & Y \arrow[d, "y"] \\ A' \arrow[r, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & B' \arrow[d, "g"] \\ A \arrow[r, "p"', "\shortmid" marking] & B \end{tikzcd}.$$

Restrictions in equipments

Prop

A double category is an equipment iff every
$$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, "g"] \\ A' \arrow[r, "p"', "\shortmid" marking] & B' \end{tikzcd}$$
has a restriction.

proof

$(\Rightarrow)$

$$\begin{tikzcd} A \arrow[r, "f_*", "\shortmid" marking] \arrow[dr, "f"'] \arrow[dr, phantom, "\beta", near start, shift left=3] & A' \arrow[r, "p", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "="] & B' \arrow[r, "g^*", "\shortmid" marking] \arrow[d, equal] & B \arrow[dl, "g"] \arrow[dl, phantom, "\delta", near start, shift right=3] \\ & A' \arrow[r, "p"', "\shortmid" marking] & B' & \end{tikzcd}$$

exhibit $f_* p g^*$ as a restriction.

$(\Leftarrow)$

A restriction
$$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, equal] \\ B \arrow[r, equal, "\shortmid" marking] & B \end{tikzcd}$$
realizes f_*

∎

Remark

This is also equivalent to the vertical dual condition.

$$\begin{tikzcd} A' \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{opc}"] & B' \arrow[d, "g"] \\ A \arrow[r, "\shortmid" marking] & B \end{tikzcd}$$

We now turn to define "equipments with finite products".

Cartesian equipments

Def (cartesian objects, CKW'91)

$\mathcal{K}$: 2-category with finite products

$\mathcal{C} \in \mathcal{K}$ is a cartesian object if

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{C}\times\mathcal{C} \arrow[l, bend left, "\exists\times"] \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\exists 1"] \end{tikzcd}$$

Let $\mathbb{E}\mathrm{qp}$ be the 2-cat. of equipments, double functors, and vertical transformation.

With this, we can express the universal property with respect to vertical arrows.

A vertical transformation.

$\alpha : F \Rightarrow G : \mathbb{D} \to \mathbb{E}$ consists of

$$\left(\begin{tikzcd} FA \arrow[d, "\alpha_A"'] \\ GA \end{tikzcd} \right)_{A \in \mathbb{D}} \quad \& \quad \left(\begin{tikzcd} FA \arrow[r, "Fp", "|"' marking] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\alpha_p"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gp"', "|" marking] & GB \end{tikzcd} \right)_{A \overset{p}{\to} B}$$

with the naturality condition.

Cartesian equipments

Def (cartesian objects, CKW'91)

$\mathcal{K}$: 2-category with finite products

$\mathcal{C} \in \mathcal{K}$ is a cartesian object if

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{C}\times\mathcal{C} \arrow[l, bend left, "\exists\,\times"] \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\exists\,1"] \end{tikzcd}$$

Let $\mathbb{E}\mathrm{qp}$ be the 2-cat. of equipments, double functors, and vertical transformation.

With this, we can express the universal property with respect to vertical arrows.

Def [Aleiferi '19]

A cartesian equipment is a cartesian object in $\mathbb{E}\mathrm{qp}$.

Prop (Unraveling the data) [Aleiferi '19]

An equipment $\mathbb{D}$ is cartesian iff

(i) its vertical category is cartesian,

(ii) the horizontal hom-categories $\mathbb{D}(A,B)$ are cartesian for $A, B \in \mathbb{D}$,

(iii) The lax functors induced by (i),(ii) $\mathbb{D}\times\mathbb{D} \xrightarrow{\times} \mathbb{D}$, $1 \xrightarrow{I} \mathbb{D}$ are double (= pseudo) functors.

More examples of cartesian equipments

- $\mathbb{R}\mathrm{el}(\mathcal{E})$

  $\mathcal{E}$: a regular category

- $\mathbb{S}\mathrm{pan}(\mathcal{E})$

  $\mathcal{E}$: a category w/ finite limits

- $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{E})$ [HN.'25]

  (E,M) : an SOFS on $\mathcal{E}$

$$\frac{A \overset{R}{\rightarrow\!\!\!\!\!+} B}{\overline{\overline{(R \to A \times B) \in M}}}$$

- $\mathbb{P}\mathrm{rof}(S)$ (internal profunctors)

  S : a category w/ finite limits and stable coequalizer.

- $S\text{-}\mathbb{P}\mathrm{rof}$ (enriched profunctors)

  S : a cartesian category w/ coequalizers preserved by $(A\times -)$'s.

- $\mathbb{L}\mathrm{ex}$: obj. : categories w/ finite limits

$$\begin{array}{c}\begin{tikzcd} A \arrow[r, "k", "|"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\tau"] & B \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "L"', "|"{marking}] & \mathcal{D} \end{tikzcd}\end{array} \quad \Big\| \quad \exists \begin{array}{c}\begin{tikzcd} A \arrow[r, "k"] \arrow[dr, phantom, "\Downarrow\tau"] & B \\ \mathcal{C} \arrow[u, bend right] \arrow[r, "L"'] & \mathcal{D} \arrow[u, bend right] \end{tikzcd}\end{array}$$

$F \dashv$, $G \dashv$; fin. lim. preserving functors

Merits in cartesian equipments

- Easy to check the condition

Example Rel is cartesian.

proof

$$\mathbb{R}\mathrm{el} \times \mathbb{R}\mathrm{el} \underset{\times}{\overset{\Delta}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbb{R}\mathrm{el}$$

$$\left(\begin{tikzcd} A \arrow[r, "R", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\cap\mathrm{I}"] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "|"] & D \end{tikzcd} , \begin{tikzcd} A' \arrow[r, "R'", "|"] \arrow[d, "f'"'] \arrow[dr, phantom, "\cap\mathrm{I}"] & B' \arrow[d, "g'"] \\ C' \arrow[r, "S'"', "|"] & D' \end{tikzcd}\right) \mapsto \begin{tikzcd} A \times A' \arrow[r, "R \times R'", "|"] \arrow[d, "f \times f'"'] \arrow[dr, phantom, "\cap\mathrm{I}"] & B \times B' \arrow[d, "g \times g'"] \\ C \times C' \arrow[r, "S \times S'"', "|"] & D \times D' \end{tikzcd}$$

$$\left(\begin{tikzcd} X \arrow[r, "P", "|"] \arrow[d, "s"'] \arrow[dr, phantom, "\cap\mathrm{I}"] & Y \arrow[d, "t"] \\ A \arrow[r, "R"', "|"] & B \end{tikzcd} , \begin{tikzcd} X \arrow[r, "P", "|"] \arrow[d, "s'"'] \arrow[dr, phantom, "\cap\mathrm{I}"] & Y \arrow[d, "t'"] \\ A' \arrow[r, "R'"', "|"] & B' \end{tikzcd}\right) \Bigg\| \begin{tikzcd} X \arrow[r, "P", "|"] \arrow[d, "\langle s, s' \rangle"'] \arrow[dr, phantom, "\cap\mathrm{I}"] & Y \arrow[d, "\langle t, t' \rangle"] \\ A \times A' \arrow[r, "R \times R'"', "|"] & B \times B' \end{tikzcd}$$

- Well-behaved with some constructions

The general theory of cartesian objects helps us construct a new cartesian equipments.

Example

$\mathbb{P}\mathrm{rof} = \mathbb{M}\mathrm{od}(\mathbb{S}\mathrm{pan})$ is a cartesian equipment.

proof

$\mathbb{E}\mathrm{q}\mathbb{p}^{*} \xrightarrow{\mathbb{M}\mathrm{od}} \mathbb{E}\mathrm{q}\mathbb{p}$ preserves $\times$ and 1

$\Longrightarrow$ lifts to $\mathbb{E}\mathrm{q}\mathbb{p}^{*}_{\mathrm{cart}} \xrightarrow{\mathbb{M}\mathrm{od}} \mathbb{E}\mathrm{q}\mathbb{p}_{\mathrm{cart}}$.

(We need to assume local coequalizers for equipments that we are applying $\mathbb{M}\mathrm{od}$ to. $*$ is for this condition.)

- Flexible

We no longer need to take maps as vertical arrows.

Example Maps in **Prof** need not be of form F_*.

They are "functors up to Cauchy completion".

Cartesian bicategories VS cartesian equipments

Q. Is there a cartesian equipment whose horizontal part is $\mathbb{B}$?

($\mathbb{B}$: cartesian bicategory)

A. Almost.

Verity's PhD thesis presents a double bicategorical cartesian equipment $\mathrm{Map}(\mathbb{B})$ whose horizontal part is $\mathbb{B}$.

(This is inevitable due to the coherence problem without further assumption [N. '25, Lemma 2.5.15].)

Q. Is the horizontal part of $\mathbb{D}$ a cartesian bicategory ?

($\mathbb{D}$: cartesian equipment)

A. Yes. [Patterson '24, N. '25]

Patterson '24: Abstract proof based on Trimble's definition.

N. '25: Diagrammatic proof

Remark

This theorem states that $A \times B$ in a cart. eqp. has the universal property not only among vertical arrows but also among maps of horizontal arrows.

Outline

1. Cartesian bicategories
2. Cartesian equipments
3. Double categories of relations

Double categories of relations

$=$ Double categories that serve as semantic environments for predicate logic by modeling binary relations

○	×
$\mathbb{R}\mathrm{el}(\mathcal{E})$	$\mathbb{P}\mathrm{rof}(S)$
$\mathbb{S}\mathrm{pan}(\mathcal{E})$	$S\text{-}\mathbb{P}\mathrm{rof}$
$\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{E})$	$\mathbb{L}\mathrm{ex}$

What set these apart?

"Symmetry/involution" ?

"Horizontal arrows $A \overset{P}{\nrightarrow} B$ are only dependent on $A \times B$" ?

Definition 2.1. (i) An object X in a Cartesian bicategory is *discrete* when the multiplication Δ_X^* and the comultiplication Δ_X satisfy

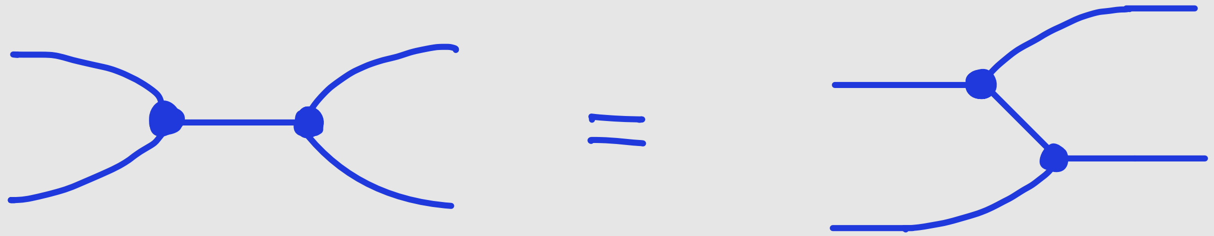
(D) $\quad \Delta \cdot \Delta^* = (\Delta^* \otimes 1) \cdot (1 \otimes \Delta).$

(We are forgetting the associativity in the middle.)

(ii) A Cartesian bicategory is called a '*bicategory of relations*' if every object is discrete.

[Carboni, Walters. '87]

This condition was later rephrased as the Frobenius law, because this is equivalent to the condition for Frobenius algebra [WW '08].



The Frobenius law for cartesian bicategories state that

$$\exists a \left(\begin{matrix} x \\ y \end{matrix} \rightsquigarrow a \rightsquigarrow \begin{matrix} z \\ w \end{matrix} \right) \iff \begin{tikzcd} x \arrow[r, squiggly] & z \\ y \arrow[ur, squiggly] \arrow[r, squiggly] & w \end{tikzcd}.$$

(When $A \overset{\mathrm{id}}{\mathrel{\mapstochar\rightarrow}} A$ is understood as $\{(a, a') \mid a \rightsquigarrow a'\}$)

The Frobenius law

The Frobenius law for bicategories / equipments appears in the literature to characterize a certain classes of those bi/double categories.

[CW '88, WW '08, LWW '10, Aleiferi '19, Lambert '23, HN '25]

What does the Frobenius law actually mean?

Prop [WW '08, HN '25, N. 25]

A Frobenius cartesian equipment always comes with a (horizontal) self-dual compact closed structure w.r.t. $\times$ on the horizontal bicategory in the sense of [Stay '13].

$$\frac{A \times B \longrightarrow\!\!\!\!\!\!\!+\;\; C}{A \longrightarrow\!\!\!\!\!\!\!+\;\; B \times C}$$

The Frobenius law and fibrations

Thm [N. 25]

A Frobenius cart. eqp. $\mathbb{D}$ induces an elementary existential fibration $\begin{array}{c} \mathcal{U}_{\mathbb{D}} \\ \downarrow \scriptstyle{\mathcal{H}_{\mathbb{D}}} \\ \mathbb{D}_0 \end{array}$.

the base category = the vertical category $\mathbb{D}_0$

the fiber over A = the category of horizontal arrows of the form $A \overset{P}{\to} 1$

and cells $\begin{array}{ccc} A & \overset{P}{\longrightarrow} & 1 \\ \| & \tau & \| \\ A & \underset{Q}{\longrightarrow} & 1 \end{array}$ between them.

cartesian lifting along $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$ = restriction $\begin{array}{ccc} A & \overset{f^*P}{\longrightarrow} & 1 \\ f\downarrow & \mathrm{cart} & \| \\ B & \underset{P}{\longrightarrow} & 1 \end{array}$

Remark What is not obvious is the Frobenius reciprocity for $\mathcal{H}_{\mathbb{D}}$.

The Frobenius law and fibrations

$\boxed{\text{Thm}}$ [N. 25]

A Frobenius cart. eqp. $\mathbb{D}$ induces an elementary existential fibration
$\begin{tikzcd} \mathcal{U}_{\mathbb{D}} \arrow[d, "H_{\mathbb{D}}"] \\ \mathbb{D}_0 \end{tikzcd}$.

$\boxed{\text{Thm}}$ [N. 25] We have an equivalence $\mathbf{Eqp}_{\mathrm{cart,Frob}} \xrightarrow[\simeq]{H} \mathbf{EEF}$.

Remark Its inverse is $\mathbf{EEF} \xrightarrow{\mathbb{B}\mathrm{il}} \mathbf{Eqp}_{\mathrm{cart}}$ in [N. 25].

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\tau"] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "|" marking] & D \end{tikzcd} \text{ in } \mathbb{B}\mathrm{il}(P) \quad \Bigg\| \quad \begin{tikzcd} R \arrow[r, "\tau"] \arrow[d, mapsto] & S \arrow[d, mapsto] \\ A \times B \arrow[r, "f \times g"'] & C \times D \end{tikzcd} \text{ in } \begin{tikzcd} \mathcal{E} \arrow[d, "P"] \\ \mathcal{B} \end{tikzcd}$$

See SideA of this talk.

Reference

- E. Aleiferi. Cartesian Double Categories with an Emphasis on Characterizing Spans. PhD thesis, Dalhousie Univ, 2018.
- A. Carboni & R.F.C. Walters. Cartesian bicategories I. JPAA, 49(1-2): 11-32, 1987.
- A. Carboni, G.M. Kelly, R.J. Wood. A 2-categorical approach to change of base and geometric morphisms. I. International Category Theory Meeting, 32: 47-95. 1991
- A. Carboni, G.M. Kelly, R.F.C. Walters, R.J. Wood. Cartesian bicategories II. TAC, 19:93-124, 2007
- K. Hoshino & H. Nasu. Double categories of relations relative to factorisation system. (ACS, 2025.)
- E. Patterson. Transposing cartesian and other structure in double categories. preprint, 2024
- M. Shulman. Framed bicategories and monoidal fibrations TAC. 20, No.18. 2008.
- T. Trimble and others. cartesian bicategory. nLab. 2009. version 27
- D. Verity. Enriched categories, internal categories and change of base. Repr. TAC. 36: No.8. 206-249, 2011.
- R.F.C. Walters, R.J. Wood. Frobenius objects in Cartesian bicategories. TAC. 20. No.3: 25-47. 2008.

Thank you!

my homepage :

hayatonasu.github.io

my thesis :

arXiv:2501.17869

Appendix: the Frobenius law in equipments

In terms of cartesian equipments, the Frobenius law can be stated as :

the pullback

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ A\times A \arrow[dr, "\Delta\times id"'] & & A\times A \arrow[dl, "id\times\Delta"] \\ & A\times A\times A & \end{tikzcd}$$

is factored as

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[d, phantom, "\text{opc.}"] & \\ A\times A \arrow[rr, "|"] \arrow[dr, "\Delta\times id"'] & \arrow[d, phantom, "\text{cart.}"] & A\times A \arrow[dl, "id\times\Delta"] \\ & A\times A\times A & \end{tikzcd},$$

(the Beck-Chevalley pullback [HN.'25].)

Elementary Existential Fib.

||

bifibrations finite products with

the Beck-Chevalley cond

\+ for Δ, π

the Frobenius reciprocity

$\xrightarrow{\text{Bil}}$

$\xleftarrow[4]{}$

Cartesian equipments

the Frobenius law