Double categories of relations relative to factorization systems

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Introduction

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}$$

sets and functions

⇝ categories

+

$$\begin{tikzcd} A \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\leq"] & B \end{tikzcd}$$

sets, relations, and the order on relations

$(R \subseteq A \times B)$

⇝ bicategories

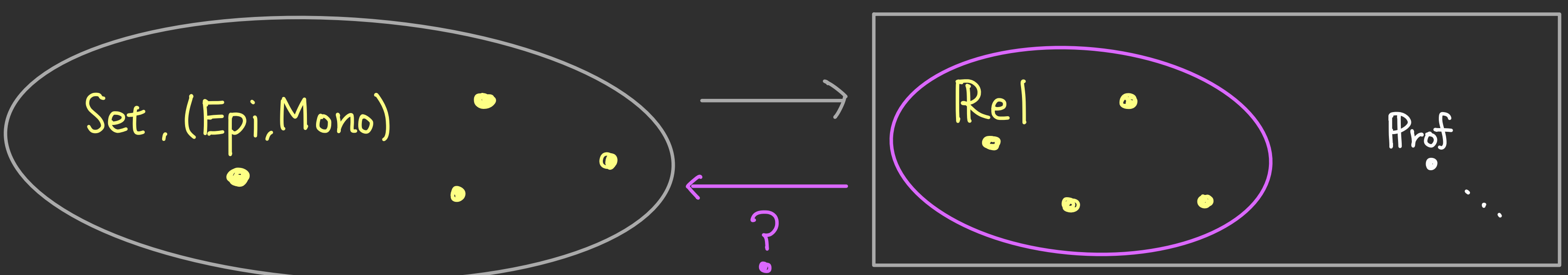
= Double categories of relations $\mathbb{R}\mathrm{el}$

$$\begin{tikzcd} A \arrow[r, "R"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "g"] \\ B \arrow[r, "S"'] & D \end{tikzcd}$$

sets, functions, relations, the order

Factorization Systems

Double Categories



Outline

1. Background : Double categories and relativized relations.
2. Structures characteristic to double categories of relations.
3. A characterization theorem and its consequences

This talk is based on

Keisuke Hoshino, Hayato Nasu. Double categories of relations relative to factorisation systems, arXiv 2310. 19428

Outline

1. Background : Double categories and relativized relations.

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Double categories of relations and spans

A (pseudo) double category is an internal pseudo category in $\mathcal{CAT}$.

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{\xrightarrow{\mathrm{cod}}}{\overset{\xrightarrow{\mathrm{dom}}}{\xleftarrow{\mathrm{Id}}}} \mathbb{D}_0 \quad \text{in } \mathcal{CAT}$$

s.t. ...

$\mathbb{D}_0$: the category of objects and vertical arrows $\bullet \downarrow \bullet$

$\mathbb{D}_1$: the category of horizontal arrows $\bullet \mathrel{\mapstochar\rightarrow} \bullet$ and cells

$$\begin{tikzcd} \bullet \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[d] \\ \bullet \arrow[r, "\shortmid" marking] & \bullet \end{tikzcd}$$

e.g., $\mathbb{P}\mathrm{rof}$, $\mathbb{T}\mathrm{opos}$, ...

	$\mathbb{R}\mathrm{el}(\mathcal{C})$ ($\mathcal{C}$: regular)	$\mathbb{S}\mathrm{pan}(\mathcal{C})$ ($\mathcal{C}$: fin. complete)
objects	objects in $\mathcal{C}$	
v-arrows	arrows in $\mathcal{C}$	
h-arrows $A \overset{R}{\mathrel{\mapstochar\rightarrow}} B$	relations $R \rightarrowtail A \times B$	spans in $\mathcal{C}$ $A \xleftarrow{l_R} R \xrightarrow{r_R} B$
cells $\begin{tikzcd} A \arrow[r, "R", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "\shortmid" marking] & D \end{tikzcd}$	"inclusion order" $(a,b) \in R \Downarrow (f(a), g(b)) \in S$	$\begin{tikzcd} A \arrow[d, "f"'] & R \arrow[l, "l_R"'] \arrow[r, "r_R"] \arrow[d, "\alpha"] & B \arrow[d, "g"] \\ C & S \arrow[l, "l_S"] \arrow[r, "r_S"'] & D \end{tikzcd}$

Can we unify these?

Relations relative to a factorization system

Definition [Klein'70, Kelly'91, Pavlović'95]

$\mathcal{C}$: a finitely complete category

(E, M) : a stable orthogonal factorization system (SOFS)

An M-relation $A \overset{R}{\nrightarrow} B$ in $\mathcal{C}$ is an arrow $R \xrightarrow{\langle l_R, r_R \rangle} A \times B \in M$.

$$(E, M) = \begin{cases} (\mathrm{RegEpi}, \mathrm{Mono}) & \text{on a reg. cat.} \\ (\mathrm{Iso}, \mathrm{Mor}) & \end{cases} \rightsquigarrow \quad M\text{-relations} = \begin{cases} \text{relations} \\ \text{spans} \end{cases}$$

The composite $R \odot S$ of $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ is defined as

$$\begin{tikzcd} T \arrow[r, "\langle p, q \rangle"] \arrow[dr, "E"'] & R \times S \arrow[r, "l_R \times r_S"] & A \times C \\ & R \odot S \arrow[ur, "\in M"'] & \end{tikzcd}$$

where

$$\begin{tikzcd} & & & T \arrow[dl, "p"'] \arrow[dr, "q"] & & & \\ & & R \arrow[dl, "l_R"'] \arrow[dr, "r_R"] & & S \arrow[dl, "l_S"'] \arrow[dr, "r_S"] & & \\ & A & & B & & C & \end{tikzcd}$$

Double categories of relative relations

Definition [Hoshino-N.]

For an SOFS (E, M) on $\mathcal{C}$,

$\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is defined as :

objects : objects in $\mathcal{C}$,

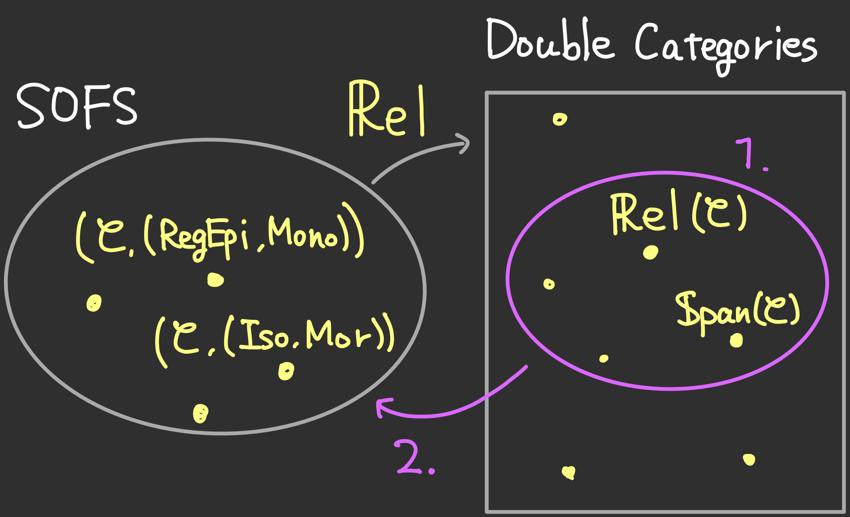
vertical arrows : arrows in $\mathcal{C}$,

horizontal arrows : M-relations,

cells
$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "|"{marking}] & D \end{tikzcd}
\quad : \quad
\begin{tikzcd} & R \arrow[dl, "l_R"'] \arrow[dr, "r_R"] \arrow[d] & \\ A \arrow[d, "f"'] & S \arrow[dl, "l_S"] \arrow[dr, "r_S"'] & B \arrow[d, "g"] \\ C & & D \end{tikzcd}$$

Goal

1. Find the conditions that characterize the double categories of relations.
2. Recover the factorization system from the double category.



Characterization of double categories of relations / spans

The characterization has already been done for the special cases.

$\boxed{\text{Theorem}}$ [Ale '18]

$\mathbb{D} \simeq \mathbb{S}\mathrm{pan}(\mathcal{C})$ for some category $\mathcal{C}$ with finite limits

if and only if $\mathbb{D}$ is a unit-pure equipment with strong Eilenberg-Moore objects for horizontal copointed endomorphisms, and ***.

$\boxed{\text{Theorem}}$ [Lam '22]

$\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{C})$ for some regular category $\mathcal{C}$

if and only if $\mathbb{D}$ is a locally posetal, discrete, cartesian equipment with subobject comprehension scheme.

Can we generalize these results with SOFS's ?

Outline

1. Background : Double categories and relativized relations.

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Notational remark

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ B \arrow[rr, "R"', "|" marking] & & C \end{tikzcd} \quad \text{stands for} \quad \begin{tikzcd} A \arrow[r, "\mathrm{Id}_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "g"] \\ B \arrow[r, "R"', "|" marking] & C \end{tikzcd}.$$

Equipments

A **restriction** of $\begin{tikzcd} \cdot \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "R"', "|" marking] & \cdot \end{tikzcd}$ is a cell $\begin{tikzcd} \cdot \arrow[r, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\rho"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "R"', "|" marking] & \cdot \end{tikzcd}$ s.t.

$$\begin{tikzcd} \cdot \arrow[r, "S", "|" marking] \arrow[d, "h"'] \arrow[ddr, phantom, "\forall\, \xi"] & \cdot \arrow[d, "k"] \\ \cdot \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "R"', "|" marking] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "S", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\exists!\, \tilde{\xi}"] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\rho"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "R"', "|" marking] & \cdot \end{tikzcd}$$

An **extension** of $\begin{tikzcd} \cdot \arrow[r, "S", "|" marking] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot & \cdot \end{tikzcd}$ is a cell $\begin{tikzcd} \cdot \arrow[r, "S", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\lambda"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "|" marking] & \cdot \end{tikzcd}$ s.t. ... .

The cells ρ and λ are written as $\begin{tikzcd} \cdot \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "|" marking] & \cdot \end{tikzcd}$ and $\begin{tikzcd} \cdot \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\text{opc}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "|" marking] & \cdot \end{tikzcd}$.

$\boxed{\text{Lem}}$ Every $\begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "|" marking] & \cdot \end{tikzcd}$ has a restricsion iff every $\begin{tikzcd} \cdot \arrow[r, "|" marking] \arrow[d] & \cdot \arrow[d] \\ \cdot & \cdot \end{tikzcd}$ has an extention.

A double category $\mathbb{D}$ is an **equipment** if these equivalent conditions hold.

e.g., $\mathbb{P}\mathrm{rof}$ is an equipment.

$\boxed{\text{Lem}}$ $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is an equipment.

$\boxed{\text{Restriction is substitution of functions into a relation.}}$

Proof

$\begin{tikzcd} A \arrow[r, "{R(f,g)}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart}"] & B \arrow[d, "g"] \\ C \arrow[r, "R"', "|" marking] & D \end{tikzcd}$ is given by $\begin{tikzcd} \bullet \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \times B \arrow[d, "f \times g"] \\ R \arrow[r, tail] & C \times D \end{tikzcd}$ ∎

Tabulator

A tabulator of $A \overset{R}{\nrightarrow} B$ is a cell

$$\begin{tikzcd} & T \arrow[dl, "l"'] \arrow[dr, "r"] & \\ A \arrow[rr, "|"{marking}, "R"'] \arrow[rr, phantom, bend left=30, "\kappa"] & & B \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "|"{marking}, "R"'] \arrow[rr, phantom, bend left=30, "\forall\xi"] & & B \end{tikzcd} = \begin{tikzcd} & X \arrow[d, "\exists!\,\tilde{\xi}"] & \\ & T \arrow[dl, "l"'] \arrow[dr, "r"] & \\ A \arrow[rr, "|"{marking}, "R"'] \arrow[rr, phantom, bend left=30, "\kappa"] & & B \end{tikzcd} .$$

[Grandis. Paré, '99]

If κ is opcartesian, we call it a strong tabulator.

$\boxed{\text{Lem}}$ $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ has strong tabulators for all the horizontal arrows.

Proof An M-relation $R \overset{\langle l,r\rangle}{\rightarrowtail} A \times B$ comes with

$$\begin{tikzcd} & R \arrow[dl, "l"'] \arrow[dr, "r"] & \\ A \arrow[rr, "|"{marking}, "R"'] \arrow[rr, phantom, bend left=30, "\bullet"] & & B \end{tikzcd}$$

∎

$\boxed{\text{Tabulator is comprehension of relations.}}$

Beck-Chevalley pullbacks

$$\begin{tikzcd} \mathbb{D}_1 \arrow[d, "\langle \mathrm{cod}, \mathrm{dom} \rangle"] \\ \mathbb{D}_0 \times \mathbb{D}_0 \end{tikzcd}$$

A pullback square
$$\begin{tikzcd} & D \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner" rotate=-45] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$
in Set

$$\rightsquigarrow \{ \underbrace{(p(d), q(d)) \mid d \in D}_{\text{extension of } p \text{ and } q} \} = \{ \underbrace{(a, b) \mid f(a) = g(b)}_{\text{restriction of } f \text{ and } g} \} \subseteq A \times B.$$

This can be described as

$$(\star) \quad \begin{tikzcd} & D \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A \arrow[dr, "f"'] \arrow[rr, phantom, "\mathrm{Id}"] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} = \begin{tikzcd} & D \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\text{opc.}"] & \\ A \arrow[rr, "|"] \arrow[dr, "f"'] & \arrow[d, phantom, "\text{cart.}"] & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \left(= \begin{tikzcd} D \arrow[r, equal, "|"] \arrow[d, "p"'] \arrow[dr, phantom, "\text{opc.}"] & D \arrow[d, "q"] \\ A \arrow[r, "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{cart.}"] & B \arrow[d, "g"] \\ C \arrow[r, equal, "|"] & C \end{tikzcd} \right)$$

In general, a double category has **Beck-Chevalley pullbacks** if for any pullback square
$$\begin{tikzcd} & D \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner" rotate=-45] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$
in $\mathbb{D}_0$, we have $(\star)$.

Technique used in the proofs

Sandwich lemma

$$\begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{opc.}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\shortmid"{marking}] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{opc.}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{opc.}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\text{opc.}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "\shortmid"{marking}] & \cdot \end{tikzcd}$$

Toy Proposition

In an equipment $\mathbb{D}$ with Beck-Chevalley pullbacks, the class of horizontal arrows

$$\left\{ A \xrightarrow{R} B \;\middle|\; \exists \begin{tikzcd} & \cdot \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\text{opc}"] & \\ \cdot \arrow[rr, "\shortmid"{marking}, "R"'] & {} & \cdot \end{tikzcd} \right\} \text{ is}$$

closed under composition.

Proof

$$\begin{tikzcd} & & Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{opc.}"] & & \\ & X \arrow[rr, "\shortmid"{marking}] \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\text{opc.}"] \arrow[drr, phantom, "\text{cart.}"] & {} & Y \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[d, phantom, "\text{opc.}"] & \\ A \arrow[rr, "\shortmid"{marking}, "R"'] & {} & B \arrow[rr, "\shortmid"{marking}, "S"'] & {} & C \end{tikzcd}$$

$$= \begin{tikzcd} & Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{opc.}"] & \\ X \arrow[rr, "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\text{opc.}"] & {} & Y \arrow[d, "k"] \\ A \arrow[rr, "\shortmid"{marking}, "R \odot S"'] & & C \end{tikzcd}$$

$$= \begin{tikzcd} & Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{opc.}"] & \\ A \arrow[rr, "\shortmid"{marking}, "R \odot S"'] & {} & C \end{tikzcd}$$

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1. Background : Double categories and relativized relations.

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Characterization theorem

$\boxed{\text{Main Theorem}}$ [Hoshino - N.]

For a double category $\mathbb{D}$, the following are equivalent.

(i) $\mathbb{D}$ is equivalent to $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$

for some finitely complete category $\mathcal{C}$ and an SOFS (E, M) on it.

(ii) • $\mathbb{D}$ is a cartesian equipment.

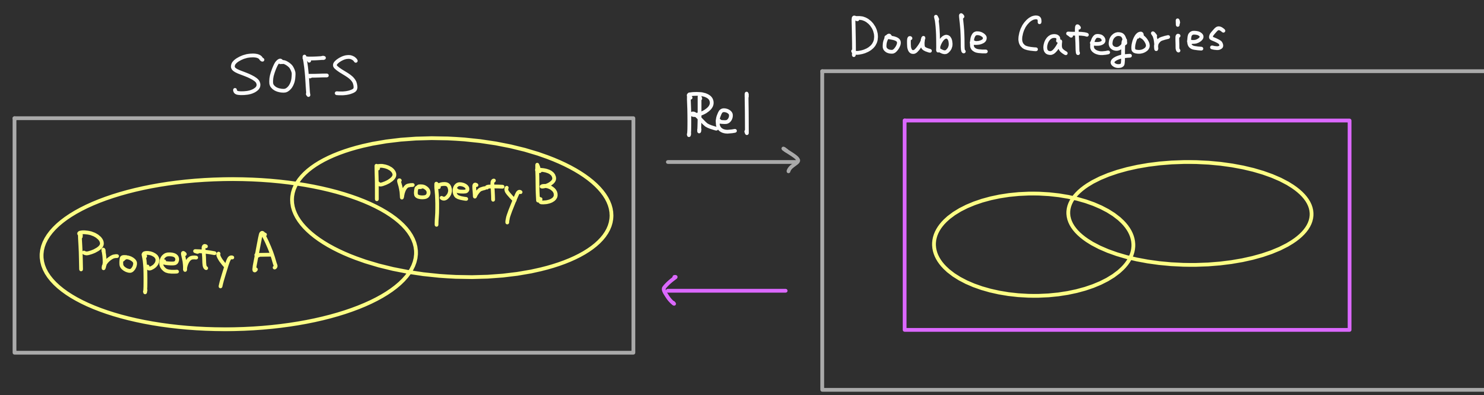
• $\mathbb{D}$ has strong tabulators and Beck-Chevalley pullbacks.

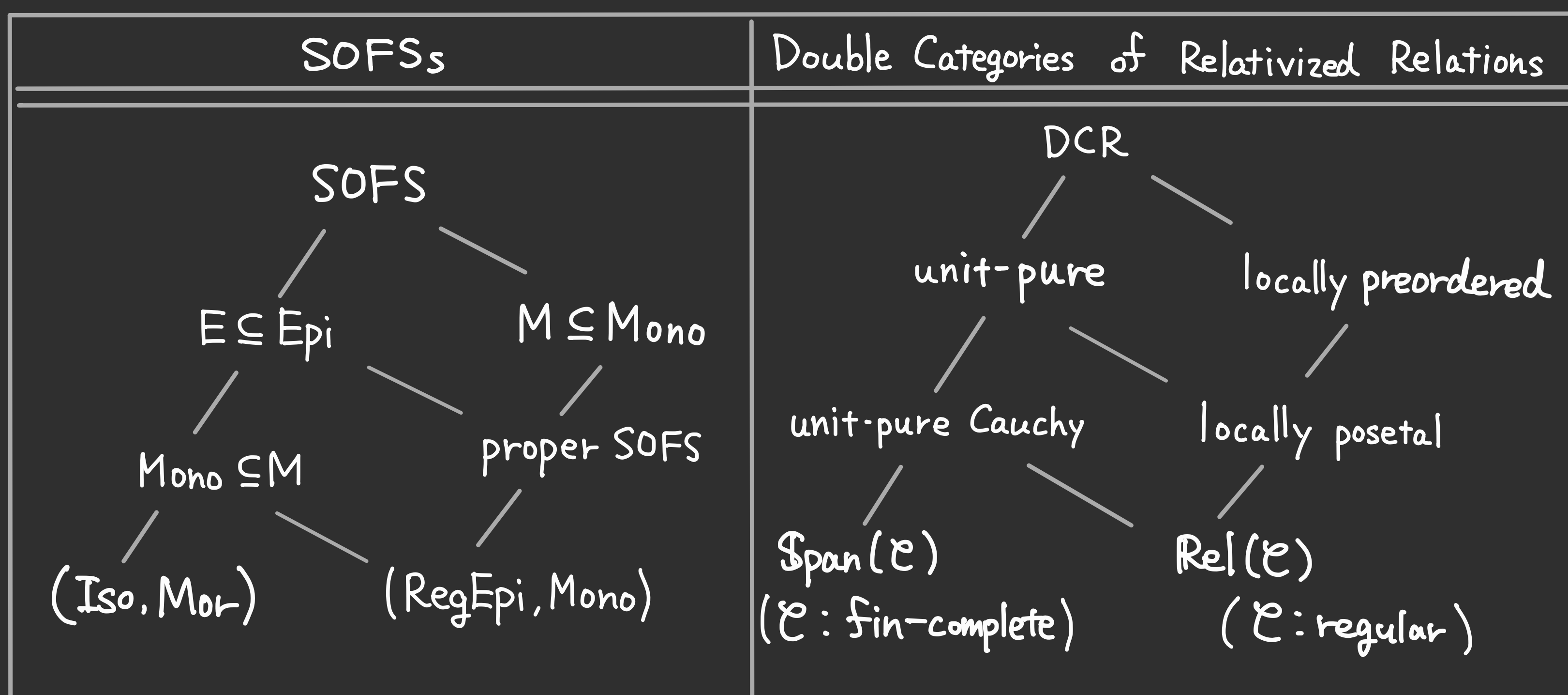
• $$M(\mathbb{D}) := \left\{ \begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ B \end{array} \;\middle|\; \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "!"] & \\ B \arrow[rr, "\exists R"', "|" marking] & & 1 \end{tikzcd} : \text{a tabulator of } R \right\}$$

is closed under composition.

If these hold, M is "the same" as $M(\mathbb{D})$.

Characterization theorem





Classes of double category

- $\mathbb{D}$ is unit-pure [Ale '18] if every cell $\begin{tikzcd} \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \arrow[d, "g"] \\ \arrow[r, equal] & \end{tikzcd}$ must be $\begin{tikzcd} \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "="] & \arrow[d, "f"] \\ \arrow[r, equal] & \end{tikzcd}$.

$\boxed{\text{Proposition}}$ [HN.] $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is unit-pure $\iff$ $E \subseteq \mathrm{Epi}$.

- $\mathbb{D}$ is locally preordered if there is at most one cell for each frame.

$\boxed{\text{Proposition}}$ [HN.] $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is locally preordered $\iff$ $M \subseteq \mathrm{Mono}$.

- $\mathbb{D}$ is Cauchy [Paré, '21] if every $\begin{tikzcd} A \arrow[r, bend left, "R"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "S"] \end{tikzcd}$ is of form $\begin{tikzcd} A \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^*"] \end{tikzcd}$.

$\boxed{\text{Proposition}}$ [HN.] $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is unit-pure Cauchy $\iff$ $\mathrm{Mono} \subseteq M$.

$\Downarrow$ recover

$\boxed{\text{Thm}}$ [Lambert '22] $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$: unit-pure, Cauchy, locally preordered

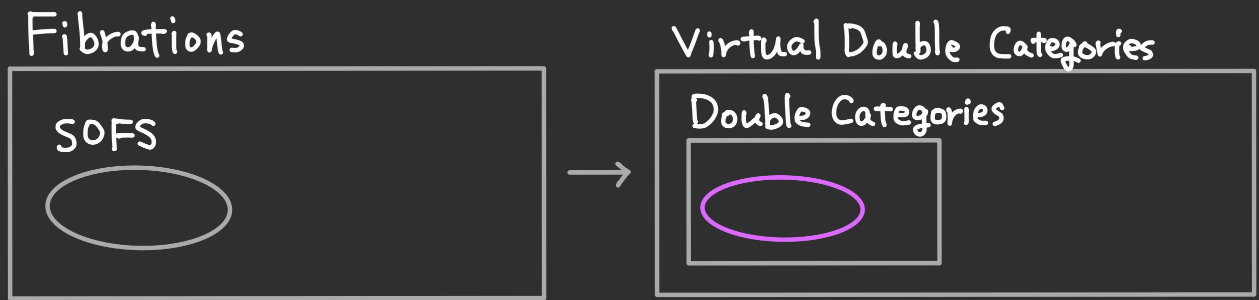
$\Rightarrow (E, M) = (\mathrm{RegEpi}, \mathrm{Mono})$, $\mathcal{C}$: regular.

Future work

- Allegories as double categories

  $\boxed{\text{BC pullbacks + locally preorderedness}} \Longrightarrow \boxed{\text{the modular law}} \Longrightarrow ?$

- Connection to Hyperdoctrines (ongoing)



  (Relational doctrines [Dagnino.Pasquali, '23])

- Would-be double categorical logic

  (Internal language of double categories (ongoing))

⋮

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Thank you!

Hayato Nasu

<table>
<tr><th>SOFSs</th><th>Double Categories of Relativized Relations</th></tr>
<tr><td>

$$
\begin{tikzcd}
 & & & \mathrm{SOFS} \arrow[dl, dash] \arrow[dr, dash] & \\
 & & E \subseteq \mathrm{Epi} \arrow[dl, dash] \arrow[dr, dash] & & M \subseteq \mathrm{Mono} \arrow[dl, dash] \\
 & \mathrm{Mono} \subseteq M \arrow[dl, dash] \arrow[dr, dash] & & \text{proper SOFS} \arrow[dl, dash] & \\
 (\mathrm{Iso}, \mathrm{Mor}) & & (\mathrm{RegEpi}, \mathrm{Mono}) & &
\end{tikzcd}
$$

</td><td>

$$
\begin{tikzcd}
 & & & \mathrm{DCR} \arrow[dl, dash] \arrow[dr, dash] & \\
 & & \text{unit-pure} \arrow[dl, dash] \arrow[dr, dash] & & \text{locally preordered} \arrow[dl, dash] \\
 & \text{unit-pure Cauchy} \arrow[dl, dash] \arrow[dr, dash] & & \text{locally posetal} \arrow[dl, dash] & \\
 \begin{array}{c} \mathbb{S}\mathrm{pan}(\mathcal{C}) \\ (\mathcal{C}: \text{fin-complete}) \end{array} & & \begin{array}{c} \mathbb{R}\mathrm{el}(\mathcal{C}) \\ (\mathcal{C}: \text{regular}) \end{array} & &
\end{tikzcd}
$$

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A few words on classical results

Another equivalent condition to be of form $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is :

$\mathbb{D}$ is a cartesian equipment with Beck-Chevalley pullbacks that admits an M-comprehension scheme for some stable system M.

$$\boxed{\begin{tikzcd} \mathbb{D}(A,B) \arrow[r, shift right, "\text{tab.}"'] \arrow[r, phantom, "\perp"] & M\text{-}\mathrm{Rel}(A,B) \arrow[l, shift right, "\text{opc.}"'] \end{tikzcd} \text{ is an equivalence.}}$$

In unit-pure double categories,

co-Eilenberg-Moore objects of horizontal comonads can replace tabulators.

tabulators of horizontal arrows	$\begin{tikzcd} \mathbb{D}(A,B) \arrow[r, shift right, "\text{tab.}"'] \arrow[r, phantom, "\perp"] & M\text{-}\mathrm{Rel}(A,B) \arrow[l, shift right, "\text{opc.}"'] \end{tikzcd}$
co-EMs of $\left\{\begin{array}{l}\text{horizontal comonads}\\ \text{horizontal copointed arrows}\end{array}\right.$	$\begin{tikzcd} \mathbb{C}\mathrm{omon}(A) \\ (\mathbb{C}\mathrm{op}(A)) \end{tikzcd} \begin{tikzcd} {} \arrow[r, shift right, "\text{coEM}"'] \arrow[r, phantom, "\perp"] & {} \arrow[l, shift right] \end{tikzcd} M/A \subseteq \mathbb{D}_0/A$

$\rightsquigarrow$ Characterization of $\mathbb{S}\mathrm{pan}$ ([Aleiferi, '18]).

Functionally completeness in literature

- Carboni and Walter's "Cartesian bicategories I"

For any $X \overset{R}{\nrightarrow} 1$, there exist
$$\begin{tikzcd} X_R \arrow[d, "f"'] \\ X \end{tikzcd}$$
s.t.
$$\begin{tikzcd} & X_R \arrow[dl, "f"'] \arrow[dr, "!"] \arrow[d, phantom, "\mathrm{opc.}"] & \\ X \arrow[rr, "\shortmid" marking] & & 1 \end{tikzcd}$$

- Lambert's "Double categories of relations"

Functionally completeness $\doteqdot$ Mono-comprehension scheme

. unit-pure + discrete $\Longrightarrow$ BC p.b.

. discrete $\Rightarrow$ ⌜unit-pure + locally preordered $\Longleftrightarrow$ locally posetal⌟

$\boxed{\text{Theorem}}$ [Lam '22]

$\rightsquigarrow$ $\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{C})$ for some regular category $\mathcal{C}$

if and only if $\mathbb{D}$ is a locally posetal, discrete, cartesian equipment

with subobject comprehension scheme ($\doteqdot$ Mono comprehension scheme)

Cauchy, unit-pure double categories of relations

Lem [Kelly'91, HN.] If $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is unit-pure,
a horizontal left adjoint has the form $A \overset{e}{\twoheadleftarrow\!\!\!<} A' \xrightarrow{f} B$ $(e \in E \cap \mathrm{Mono})$.

Proposition [HN.] $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ is unit-pure Cauchy $\iff$ $\mathrm{Mono} \subseteq M$.

Sketch of proof of $\Leftarrow$ $\quad e \in E \cap \mathrm{Mono} \overset{\text{Lem}}{\subseteq} E \cap M = \mathrm{Iso}$. ▨

- Pavlović's "Maps I: relative to factorization systems"

$$\begin{array}{ccc} \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C}) \text{ is Cauchy, unit-pure} & \overset{}{\underset{\text{our result}}{\Longleftrightarrow}} & \mathrm{Mono} \subseteq M \\ \Updownarrow \text{ by Pavlović} & & \\ E \subseteq \mathrm{RegEpi} & & \end{array}$$

There is a direct proof!

History

Relations | **Spans**

"Bicategories of spans and relations"
Carboni, Kasangian, Street 1984

Cartesian bicategories of relations
(Carboni, Walters 1987)

Cartesian bicategories of spans
(Lack, Walters, Wood 2010)

Cartesian double categories of spans (Aleiferi 2018)

Cartesian double categories of relations (Lambert 2022)

$\boxed{\text{Theorem}}$ [Lam '22]

$\mathbb{D} \simeq \mathbb{R}\mathrm{el}(\mathcal{E})$ $\quad ({}^{\exists}\mathcal{E} : \text{regular})$

$\iff \mathbb{D}$ is ***.

$\boxed{\text{Theorem}}$ [Ale '18]

$\mathbb{D} \simeq \mathbb{S}\mathrm{pan}(\mathcal{E})$ $\quad ({}^{\exists}\mathcal{E} : \text{with finite limits})$

$\iff \mathbb{D}$ is ***.