Exploring double categories of relations

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Introduction

$$\boxed{\text{Fibrations}} \xrightarrow{\mathbb{B}\mathrm{il}} \boxed{\text{Double categories}}$$

$$\mathrm{Set}^{\subseteq} \xrightarrow{\mathrm{cod}} \mathrm{Set} \qquad \mathbb{R}\mathrm{el}$$

$$\mathrm{Set}^{\to} \xrightarrow{\mathrm{cod}} \mathrm{Set} \qquad \mathbb{S}\mathrm{pan}$$

$$\mathcal{C}^{\rightarrowtail} \xrightarrow{\mathrm{cod}} \mathcal{C} \qquad \longmapsto \qquad \mathbb{R}\mathrm{el}(\mathcal{C})$$

$$Q\text{-}\mathrm{Fam} \xrightarrow{|\cdot|} \mathrm{Set} \qquad Q\text{-}\mathbb{R}\mathrm{el}$$

$$M \hookrightarrow \mathcal{C}^{\to} \xrightarrow{\mathrm{cod}} \mathcal{C} \qquad \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$$

The guiding question :

What makes a double category of relations ?

✓ $\mathbb{R}\mathrm{el}$, $\mathbb{R}\mathrm{el}(\mathcal{C})$, $\mathbb{S}\mathrm{pan}$, $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$, $\mathcal{Q}$-$\mathbb{R}\mathrm{el}$, . . .

✗ $\mathbb{P}\mathrm{rof}$, $\mathbb{P}\mathrm{ar}$, . . .

My answer : the Frobenius axiom

1. Background
2. The Frobenius axiom
3. Bonus : Maps revisited

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Bicategories of relations

Allegory

[Freyd, Scedrov, '90]

Formulation in terms of involution.

$$(-)^{\circ} : \mathcal{B}(I,J) \longrightarrow \mathcal{B}(J,I)$$

Cartesian bicategory

[Carboni, Walters '87, Carboni, et.al. '07, Todd, nLab post.]

Formulation using adjoints (= maps) in bicategories.

Theorem.

$\mathcal{B}$ is a unital tabular allegory $\iff$ $\mathcal{B} \simeq \mathrm{Rel}(\mathcal{C})$ for a regular category $\mathcal{C}$ $\iff$ $\mathcal{B}$ is a functionally-complete discrete locally-posetal cartesian bicategory.

Bicategories of relations, generalized

Definition. $p : \mathcal{E} \to \mathcal{B}$: fibration, $\mathcal{B}$ has finite products, $I, J \in \mathcal{B}$

A p-relation $I \nrightarrow J$ is an object $R \in \underbrace{\mathcal{E}_{I \times J}}$.

the fiber over $I \times J$

Proposition. [Pavlovic '96, Lawler '15]

$p : \mathcal{E} \to \mathcal{B}$: a regular fibration $\rightsquigarrow$ $\mathcal{R}_p$: a bicategory of p-relations.

Existence of a (partial) right adjoint is stated without proof.

Theorem. [Bonchi, Santamaria, Seeber, Sobocinski '21]

$$\left(\begin{array}{c}\text{elementary existential} \\ \text{doctrines}\end{array}\right) \underset{\perp}{\overset{\mathcal{R}\text{-}}{\rightleftarrows}} \left(\begin{array}{c}\text{'bicategories of} \\ \text{relations' in [CW '87]}\end{array}\right)$$

Double categories

Definition.

A (pseudo) double category consists of

- objects $A, B, \ldots$
- tight arrows $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$, ...
- loose arrows $\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] & B \end{tikzcd}$, ...
- cells $\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$, ...

with compositions of

tight arrows, $\begin{tikzcd} A \arrow[d, "f"] \\ B \arrow[d, "g"] \\ C \end{tikzcd}$

loose arrows, $\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] & K \arrow[r, "m", "\shortmid"{marking}] & X \end{tikzcd}$

and cells

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & K \arrow[r, "m", "\shortmid"{marking}] \arrow[d, "s"] \arrow[dr, phantom, "\nu"] & X \arrow[d, "u"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & L \arrow[r, "n"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & K \arrow[d, "s"] \\ B \arrow[r, "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\tau"] & L \arrow[d, "t"] \\ C \arrow[r, "r"', "\shortmid"{marking}] & M \end{tikzcd}$$

subject to some coherence conditions.

Double categories

Example	$\mathbb{R}\mathrm{el}$	$\mathbb{S}\mathrm{pan}$	$\mathbb{P}\mathrm{rof}$
Objects	sets	sets	categories
Tight arrows	functions	functions	functors
Loose arrows	relations	spans	profunctors

Cells

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\tau"] & B \arrow[d, "g"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

$\mathbb{R}\mathrm{el}$: A cell exists iff

$$p(a,b) \Downarrow q(f(a), g(b)) \quad (\forall a, b)$$

$\mathbb{S}\mathrm{pan}$:

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & {|p|} \arrow[l] \arrow[r] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "g"] \\ C & {|q|} \arrow[l] \arrow[r] & D \end{tikzcd}$$

$\mathbb{P}\mathrm{rof}$:

$$\begin{tikzcd} A^{op} \times B \arrow[dr, "p"] \arrow[d, "f^{op} \times g"'] \arrow[dr, phantom, "\Downarrow \tau"'] & \\ C^{op} \times D \arrow[r, "q"'] & \mathrm{Set} \end{tikzcd}$$

Remark

The loose composition is required to be associative only up to isomorphism.

e.g. the composition of spans

$$\begin{tikzcd} & & {|p \circ q|} \arrow[dl] \arrow[dr] & & \\ & {|p|} \arrow[dl] \arrow[dr] & & {|q|} \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

Motivating double categories

$$\begin{tikzcd}
\boxed{\text{Regular categories, Fibrations}} \arrow[rr, shift left, "\text{Rel}"] \arrow[dr] & & \boxed{\text{Bicategories}} \arrow[ll, shift left, "\text{Map}"] \arrow[dl] \\
& \boxed{\text{Double categories}} &
\end{tikzcd}$$

where functions live

where relations live

where both functions and relations can live!

① can express the interaction b/w functions & relations with companions & conjoints.

② can distinguish functions from maps $\left(\doteqdot \begin{array}{c}\text{functional} \\ \text{relations}\end{array} \right)$

Motivating double categories

③ Cartesian double categories are easier to handle.

Definition. $\mathbb{D}$: double category

$\mathbb{D}$ is cartesian if $\mathbb{D} \underset{\exists\, 1}{\overset{!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{1}$ & $\mathbb{D} \underset{\exists\, \times}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{D}\times\mathbb{D}$ in $\mathbb{D}\mathrm{bl}$.

Proposition. [Patterson '24, N.]

$\mathbb{D}$: cartesian equipment $\Longrightarrow$ $\mathcal{L}(\mathbb{D})$: a cartesian bicategory

Remark Most of known examples of cartesian bicategories arise in this way. For a cartesian bicategory $\mathcal{B}$, there is a cartesian double bicategory whose loose part is $\mathcal{B}$. [Verity '92]

1. Background
2. The Frobenius axiom
3. Bonus : Maps revisited

The Frobenius axiom

Definition 2.1. (i) An object X in a Cartesian bicategory is *discrete* when the multiplication Δ_X^* and the comultiplication Δ_X satisfy

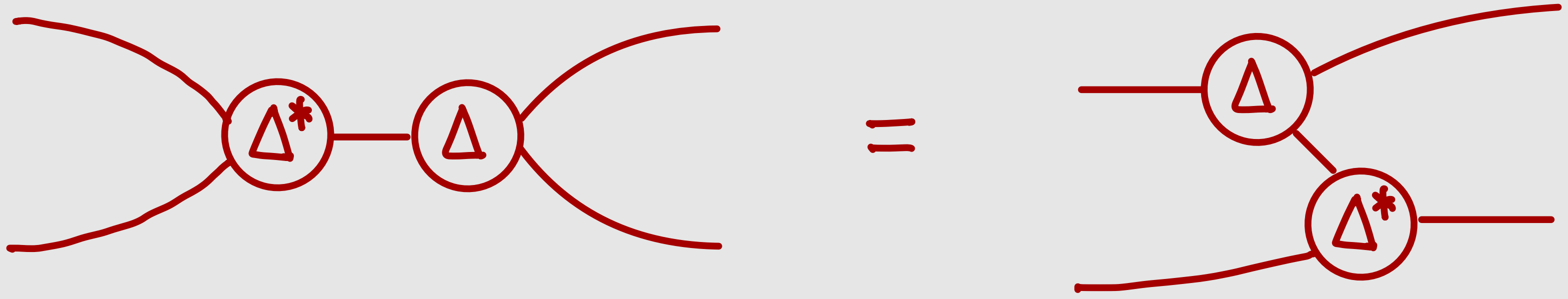
$$\text{(D)} \qquad \Delta \cdot \Delta^* = (\Delta^* \otimes 1) \cdot (1 \otimes \Delta).$$

(We are forgetting the associativity in the middle.)

(ii) A Cartesian bicategory is called a '*bicategory of relations*' if every object is discrete.

[Carboni, Walters. '87]

This condition was later rephrased as the Frobenius axiom, because this is equivalent to the condition for Frobenius algebra. [Walters, Wood '08]



The Frobenius axiom

Definition. [Walters, Wood '08, Lambert '23, Hoshino, N. '25, N.]

$\mathbb{D}$: cartesian equipment

$I \in \mathbb{D}$ is Frobenius if

$$\begin{tikzcd}[column sep=small] & I \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ I \times I \arrow[rr, "\shortmid"{marking}] \arrow[dr, "\Delta \times \mathrm{id}"'] \arrow[rr, phantom, shift left=3ex, "\text{opc.}"] \arrow[rr, phantom, shift right=3ex, "\text{cart.}"] & & I \times I \arrow[dl, "\mathrm{id} \times \Delta"] \\ & I \times I \times I & \end{tikzcd} = \begin{tikzcd}[column sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] \arrow[rr, phantom, "="] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd} \iff \left(\begin{tikzcd} I \times I \arrow[r, "\Delta^*", "\shortmid"{marking}] \arrow[d, equal] & I \arrow[r, "\Delta_*", "\shortmid"{marking}] \arrow[d, phantom, "\cong"] & I \times I \arrow[d, equal] \\ I \times I \arrow[r, "(\Delta \times \mathrm{id})_*"', "\shortmid"{marking}] & I \times I \times I \arrow[r, "(\mathrm{id} \times \Delta)^*"', "\shortmid"{marking}] & I \times I \end{tikzcd} \text{ canonically.} \right)$$

$\mathbb{D}$ is Frobenius if every object in $\mathbb{D}$ is Frobenius.

Example. $\mathcal{C}$: a regular category $\Longrightarrow$ $\mathbb{R}\mathrm{el}(\mathcal{C})$ is Frobenius

For $A \in \mathcal{C}$ and $(x, y) : A \times A$, $(z, w) : A \times A$,

$$\exists u : A \, \big((x,y) = (u,u) \wedge (u,u) = (z,w) \big) \longleftrightarrow (x, x, y) = (z, w, w)$$

Example. In $\mathbf{Prof}$, $\mathcal{C}$ is Frobenius $\Leftrightarrow$ $\mathcal{C}$ is a groupoid.

Proof sketch [Walters, Wood '08]

$\boxed{I \in \mathbb{D} \text{ is Frobenius if } \Delta^*\Delta_* \cong (\Delta\times id)_*(id\times\Delta)^*}$

$$(\Delta^*\Delta_*)((c_1,c_2),(d_1,d_2)) = \int^{e\in\mathcal{C}} \mathcal{C}(c_1,e)\times\mathcal{C}(c_2,e)\times\mathcal{C}(e,d_1)\times\mathcal{C}(e,d_2)$$

$$\downarrow$$

$$((\Delta\times id)_*(id\times\Delta)^*)((c_1,c_2),(d_1,d_2)) = \mathcal{C}(c_1,d_1)\times\mathcal{C}(c_1,d_2)\times\mathcal{C}(c_2,d_2)$$

$$\left[\begin{tikzcd} c_1 \arrow[dr] & & d_1 \\ & e \arrow[ur] \arrow[dr] & \\ c_2 \arrow[ur] & & d_2 \end{tikzcd}\right] \mapsto \left(\begin{tikzcd} c_1 \arrow[r] \arrow[dr] & d_1 \\ c_2 \arrow[r] & d_2 \end{tikzcd}\right)$$

($\Rightarrow$) Take $f : a \to b$ in $\mathcal{C}$.

$$\exists \begin{tikzcd} a \arrow[dr, "f_1"] & & a \\ & e \arrow[ur, "f_3"] \arrow[dr, "f_4"'] & \\ b \arrow[ur, "f_2"'] & & b \end{tikzcd} \mapsto \begin{tikzcd} a \arrow[r, equal] \arrow[dr, "f"] & a \\ b \arrow[r, equal] & b \end{tikzcd}.$$

$b \xrightarrow{f_2} e \xrightarrow{f_3} a$ is the inverse of f.

($\Leftarrow$) A small puzzle! ∎

The Frobenius axiom induces involution

Slogan : The Frobenius axiom is an axiom of symmetry.

Proposition. [Hoshino, N. '25, N.]

$\mathbb{D}$: a Frobenius cartesian equipment.

Every A and $\downarrow f$ in $\mathbb{D}$ are self-dual in the loose direction :

$$\mathbb{D}_\ell(C \times A, B) \cong \mathbb{D}_\ell(C, A \times B), \quad \mathbb{D}_c(h \times f, g) \cong \mathbb{D}_c(h, f \times g)$$

The unit and counit on A are

$$1 \xrightarrow{!^*} A \xrightarrow{\Delta_*} A \times A \quad \text{and} \quad A \times A \xrightarrow{\Delta^*} A \xrightarrow{!_*} 1.$$

Cor. $\mathcal{L}(\mathbb{D})$ is a compact closed bicategory in the sense of [Stay '16].

From equipments to fibrations

Definition. $\mathbb{D}$: cartesian equipment $\rightsquigarrow$

- a category $\mathcal{U}_{\mathbb{D}}$: objects : $A \xrightarrow{\alpha} 1$,

  morphisms $(A \xrightarrow{\alpha} 1) \longrightarrow (B \xrightarrow{\beta} 1)$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & 1 \arrow[d, equal] \\ B \arrow[r, "\beta"'] & 1 \end{tikzcd}$$

- a fibration $\mathcal{H}_{\mathbb{D}} : \mathcal{U}_{\mathbb{D}} \longrightarrow \mathbb{D}_0$; $(A \xrightarrow{\alpha} 1) \longmapsto A$

When $\mathbb{D}$ is Frobenius, all information in $\mathbb{D}$ is recoverable from $\mathcal{H}_{\mathbb{D}}$.

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\tau"] & A \arrow[d, "g"] \\ D \arrow[r, "\beta"'] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} C \times A \arrow[r, "\tilde{\alpha}"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\tilde{\tau}"] & 1 \arrow[d, equal] \\ D \times B \arrow[r, "\tilde{\beta}"'] & 1 \end{tikzcd} \quad \leftrightsquigarrow \quad \tilde{\alpha} \xrightarrow{\tilde{\tau}} \tilde{\beta} \text{ in } \mathcal{U}_{\mathbb{D}}$$

Proposition. [N.] $\mathbb{D}$: Frobenius $\Rightarrow$

$\mathcal{H}_{\mathbb{D}} : \mathcal{U}_{\mathbb{D}} \longrightarrow \mathbb{D}_0$ is an elementary existential fibration.

Main results

Pavlović's bicategories of predicates are extended to double categories.
[Lawler '15, N.]

$$\boxed{\text{Elementary existential fibrations}} \xrightarrow{\mathbb{B}\mathrm{il}} \boxed{\text{Cartesian equipment}}$$

$\mathbb{B}\mathrm{il}(p)$: the double category of p-relations

Theorem. [N.]

p : an e.e. fibration $\rightsquigarrow$ $\mathbb{B}\mathrm{il}(p)$: a Frobenius cartesian equipment.

This extends to a biequivalence :

$$\mathcal{E}\mathcal{E}\mathrm{Fib} \underset{\mathsf{H}}{\overset{\mathbb{B}\mathrm{il}}{\underset{\simeq}{\rightleftarrows}}} \mathrm{Cart}\mathcal{E}q_{\mathrm{Frob}}$$

$\mathbb{D} \simeq \mathbb{B}\mathrm{il}(p)$ for some p iff it is Frobenius.

Example. $\mathbb{P}\mathrm{rof}_{\mathrm{grpd}} \simeq \mathbb{B}\mathrm{il}(\mathrm{GrpdAct} \longrightarrow \mathrm{Grpd})$

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Revisiting maps

Definition. A map in a bicategory is a left adjoint 1-cell in it.

$$\begin{tikzcd}
\boxed{\text{Regular categories}} \arrow[r, shift left, "\mathrm{Rel}"] \arrow[r, phantom, "\simeq"] \arrow[d, "\mathbb{R}\mathrm{el}"'] & \boxed{\begin{array}{l}\text{Unital tabular allegories /}\\ \text{Functionally complete bicategories}\\ \text{of relations}\end{array}} \arrow[l, shift left, "\mathrm{Map}"] \\
\boxed{\text{Cartesian equipments}} \arrow[r, shift left, "\mathcal{L}(-)"] & \boxed{\text{Cartesian bicategories}} \arrow[u, phantom, "\cap" rotate=0] \arrow[l, shift left, dashed, "\mathbb{M}\mathrm{ap}"]
\end{tikzcd}$$

Definition. A bicategory $\mathcal{B}$ is map discrete if $\mathrm{Map}(\mathcal{B})(I,J) \subseteq \mathcal{B}(I,J)$

is equivalent to discrete categories.

This is sufficient to construct a double category $\mathbb{M}\mathrm{ap}(\mathcal{B})$.

Revisiting Map

Definition. [Paré 21] An equipment $\mathbb{D}$ is Cauchy if every map is isomorphic to f_* for some $f: A \to B$.

Unit-pure

$$\boxed{\begin{array}{c}\text{Map discrete}\\ \text{Cauchy cart. equipments}\end{array}} \underset{\mathbb{M}\mathrm{ap}(-)}{\overset{\mathcal{H}(-)}{\underset{\leftarrow}{\overset{\rightarrow}{\text{“}\simeq\text{”}}}}} \boxed{\begin{array}{c}\text{Map discrete}\\ \text{cart. bicategories}\end{array}}$$

Proposition. [N.]

$$\begin{tikzcd} & \mathcal{C}\mathrm{art}\mathcal{E}\mathrm{q}_{MD} \arrow[ld, "\mathbb{C}\mathrm{au}"'] \arrow[d, hook] \\ \mathcal{C}\mathrm{art}\mathcal{E}\mathrm{q}_{\mathrm{Cau}} \arrow[r, hook] \arrow[r, phantom, bend left, "\perp"] & \mathcal{C}\mathrm{art}\mathcal{E}\mathrm{q} \arrow[lu, phantom] \end{tikzcd}$$

Corollary. [Bonchi, et.al. '21, N.]

[Pavlovic, '96]

$$\mathcal{E}\mathcal{E}\mathrm{Fib} \xrightarrow[\text{locpos}]{\simeq} \mathcal{C}\mathrm{art}\mathcal{E}\mathrm{q}_{\mathrm{Frob},\cancel{MD}}^{\text{locpos}} \underset{\hookleftarrow}{\overset{\mathbb{C}\mathrm{au}}{\underset{\perp}{\rightarrow}}} \mathcal{C}\mathrm{art}\mathcal{E}\mathrm{q}_{\mathrm{Cau},\mathrm{Frob},\cancel{MD}}^{\text{locpos}} \underset{\mathbb{M}\mathrm{ap}(-)}{\overset{\mathcal{H}(-)}{\underset{\leftarrow}{\overset{\rightarrow}{\text{“}\simeq\text{”}}}}} \mathcal{C}\mathrm{art}\mathcal{B}\mathrm{i}_{\mathrm{Frob},\cancel{MD}}^{\text{locpos}}$$

Summary

$$\begin{tikzcd} \boxed{\text{Elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}", "\simeq"'] & \boxed{\boxed{\mathbb{R}\mathrm{el}(\mathcal{C}),\ \mathcal{Q}\text{-}\mathbb{R}\mathrm{el},\ \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})\ \ \text{Frobenius}}\ \ \text{Cartesian equipments}} \arrow[r, shift left=2, "\mathcal{H}(-)"] \arrow[r, phantom, "\perp"] & \boxed{\text{Cartesian bicategories}} \arrow[l, shift left=2, "\mathbb{M}\mathrm{ap}(-)"] \end{tikzcd}$$

Future Work

- Generalizing to monoidal equipments.
- Studying categorical logic with cartesian equipments (quotient completion, tabulators)

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Thank you!

my homepage :

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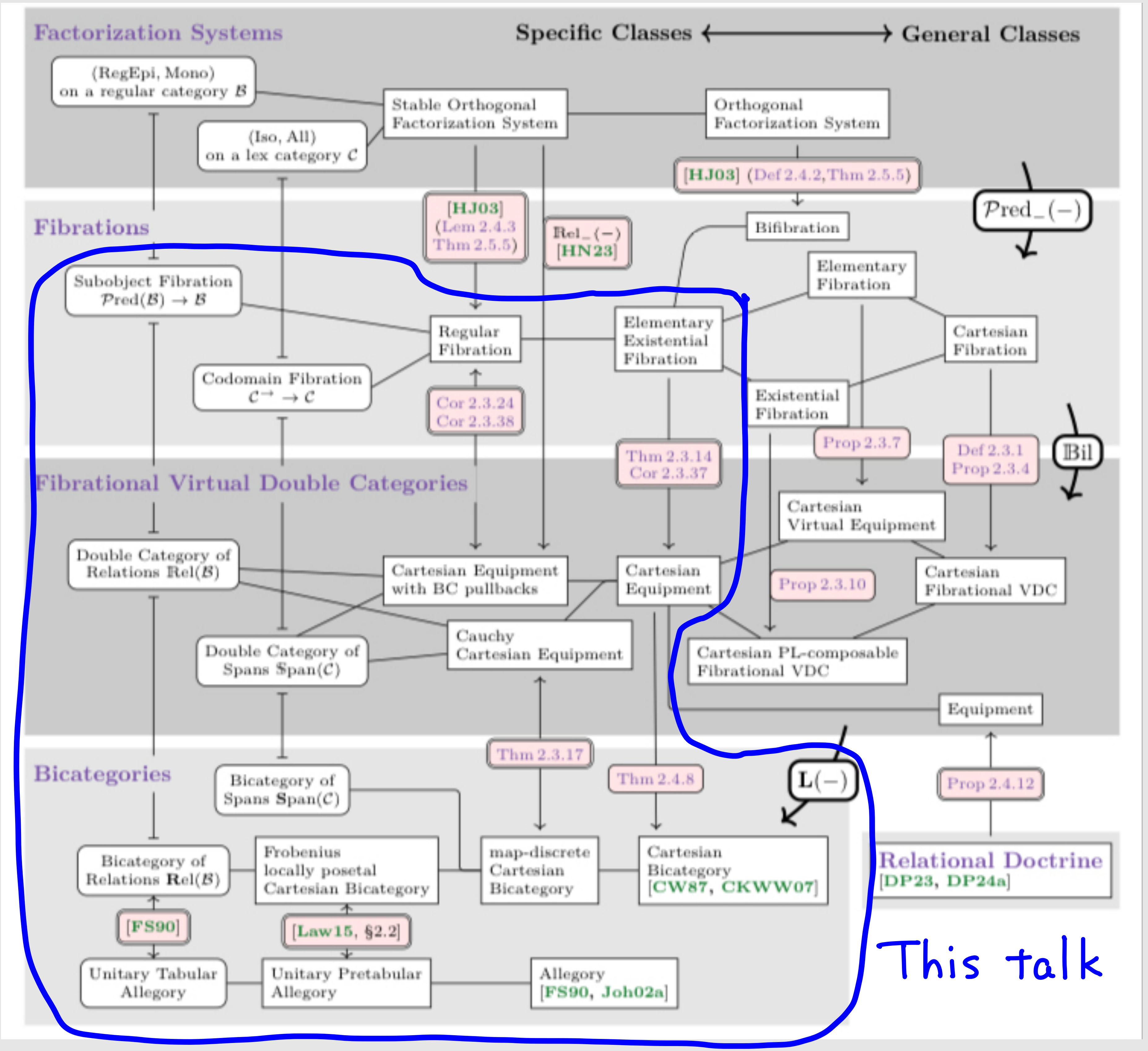
my thesis :

arXiv : 2501.17869

talk at CMS :



Characterization Results

Theorem. [Lambert '23, Hoshino.N. '25]

$\mathbb{D} \simeq \mathbf{Rel}(\mathcal{C})$ for some regular category

$\iff$ $\mathbb{D}$: locally posetal discrete cartesian equipment with Mono-comprehension scheme

Theorem. [Hoshino.N. '25, my thesis]

$\mathbb{D} \simeq \mathbf{Rel}_{(E,M)}(\mathcal{C})$ for some SOFS (E,M)

$\iff$ $\mathbb{D}$: cartesian equipment with BC pullbacks & effective tabulators and $\mathrm{Fib}(\mathbb{D})$ is closed under composition.

Bicategories of relations

A2 / A3

1984 "Bicategories of spans and relations" Carboni, Kasangian, Street.
defined Span($\mathcal{C}$) & Rel($\mathcal{C}$)

1987 "Cartesian bicategories I" Carboni, Walters.

1990 "Categories, Allegories" Freyd, Scedrov.

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defined bicategories of predicates for regular fibrations

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2023 "Double categories of relations" Lambert
characterized $\mathbb{R}\mathrm{el}(\mathcal{C})$ for regular categories $\mathcal{C}$

2025 "Double categories of relations relative to factorization systems" Hoshino, N. characterized $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$.