From Fibrations to Virtual Double Categories

Categorical Logic Meets Double Categories — Side A

Hayato Nasu

Kyoto University

hnasu@kurims.kyoto-u.ac.jp

hayatonasu.github.io

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Outline

This talk is based on my recent thesis :

Logical Aspects of Virtual Double Categories

Chapter 2 : Categorical Logic meets Double Categories

Today's slides

Fibrations (Lawvere.'70 , Jacobs.'99)

✓ terms (functions)

✗ binary relations

Today's focus

Virtual Double Categories

✓ terms (functions)

✓ binary relations

Bicategories (Carboni, Walters.'87 Freyd, Scedrov.'90)

✗ terms (functions)

✓ binary relations

1. Fibrations and Logic

2. Virtual Double Categories of Relations

3. Comparing Fibrations and Virtual Double Categories

4. Conclusion

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Taking semantics in fibrations

A functor $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$ is a (Grothendieck) fibration

$$\begin{tikzcd} \beta[u] \arrow[r, "\exists"] & \beta \\ I \arrow[r, "u"] & J \end{tikzcd}$$

s.t.

$$\begin{tikzcd} \gamma \arrow[rr] \arrow[dr, "\exists!"'] & & \beta \\ & \beta[u] \arrow[ur] & \\ K \arrow[rr] \arrow[dr, "v"'] & & J \\ & I \arrow[ur, "u"'] & \end{tikzcd} \quad \downarrow p$$

Example

(E, M) : a stable factorization system on $\mathcal{C}$

$\rightsquigarrow$ $\begin{array}{c}\mathcal{M}\\ \downarrow\\ \mathcal{C}\end{array}$ $\left(\begin{array}{l}\mathcal{M}: \text{the full subcategory of } \mathcal{C}^{\to}\\ \quad \text{spaned by } m \in M\end{array}\right)$

is a fibration.

Idea

$\mathcal{B}$ ⋯ the category of contexts Γ

and sequences of terms $\Gamma \vdash S : \Delta$

$$\begin{array}{c} x_1 : R,\ x_2 : R,\ x_3 : M \\ \downarrow (x_2 \cdot x_3,\ x_1 - x_2) \\ y_1 : M,\ y_2 : R \end{array}$$

$\mathcal{E}_\Gamma$ (the fiber category)

⋯ the category of formulas $\Gamma \vdash \varphi$

and proofs of implication $\Gamma \vdash p : \varphi \Rightarrow \psi$

$$\begin{array}{c} \exists y : R\ (y + x = y) \\ \downarrow \\ x = 0 \end{array} \qquad \mathcal{E}_{x:R}$$

Taking semantics in fibrations

Given a language $\mathcal{L}$, what structures do we need on a fibration to interpret formulas in $\mathcal{L}$?

- a sort $\sigma \in \mathcal{L}$ $\rightsquigarrow$ an object $[\![\sigma]\!]$ in $\mathbb{B}$
- a context $x_1 : \sigma_1, \dots, x_n : \sigma_n$

$\rightsquigarrow$ the product $[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$

- a function symbol $f : \sigma_1, \dots, \sigma_n \to \tau$

$\rightsquigarrow$ an arrow $[\![f]\!] : [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \longrightarrow [\![\tau]\!]$ in $\mathbb{B}$

- the interpretation of terms

is inductively defined

e.g. $[\![f(g(x), y)]\!] = [\![f]\!] \circ ([\![g]\!] \times \mathrm{id})$

- a relation symbol $R : \sigma_1, \dots, \sigma_n$

$\rightsquigarrow$ an object $[\![R]\!]$ in the fiber of $[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$

- substitution $\varphi[t_1/x_1, \dots, t_n/x_n]$

$\rightsquigarrow$

$$\begin{tikzcd} {[\![\varphi[t_1/x_1, \dots, t_n/x_n]]\!]} \arrow[r] \arrow[d, mapsto] & {[\![\varphi]\!]} \arrow[d, mapsto] \\ {[\![\tau_1]\!] \times \cdots \times [\![\tau_m]\!]} \arrow[r, "\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle"] & {[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]} \end{tikzcd}$$

using the cartesian (prone) lifting.

- the conjunction $\varphi \wedge \psi$ $\rightsquigarrow$ fiberwise product $[\![\varphi]\!] \wedge [\![\psi]\!]$
- the existential quantifier $\exists z : \sigma . \varphi(\vec{x}, z)$

$\rightsquigarrow$ the image of $[\![\varphi(\vec{x}, z)]\!] \in \mathcal{E}_{[\![\Gamma]\!] \times [\![\sigma]\!]}$

by the left adjoint

$$\begin{tikzcd} \mathcal{E}_{[\![\Gamma]\!] \times [\![\sigma]\!]} \arrow[r, bend left, "\exists_\pi"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{[\![\Gamma]\!]} \arrow[l, bend left, "(-)[\pi]"] \end{tikzcd}$$

$(\pi : [\![\Gamma]\!] \times [\![\sigma]\!] \longrightarrow [\![\Gamma]\!])$

Taking semantics in fibrations

A sufficient structure for regular logic is an elementary existential doctrine by Lawvere.

$\boxed{\text{Def}}$ A fibration p is cartesian if

- the base category $\mathbb{B}$ has finite products,
- the fiber categories $\mathcal{E}_I$ have finite products, and the reindexings $(-)[u]$ preserve them.

$\boxed{\text{Def}}$ A fibration p is elementary existential

if
- it is cartesian,
- all reindexing functors $(-)[u]$ for $\xrightarrow{u}$ of the forms $\begin{cases} I \times J \xrightarrow{\pi_2} J \\ I \times J \xrightarrow{\Delta \times id} I \times I \times J \end{cases}$ have a left adjoint $\exists_u$, and
- it satisfies some axioms.

(the Beck-Chevalley condition
the Frobenius reciprocity)

We have hierarchies of logical systems and classes of fibrations.

$\wedge, =, \exists$	elementary existential fibrations
$\wedge, =$	elementary fibrations
$\wedge$	cartesian fibrations

We will add a new column of hierarchy: the hierarchy of virtual double categories.

Constructing a new fibration

⚠ Fibers are pre-ordered in this slide.

Quotient completion (Maietti, Rosolini. '13)

The quotient completion is an analogue of the exact completion in the context of doctrines.

$$\begin{array}{ccc} \mathcal{E} & & \mathcal{D}es(p) \\ {\scriptstyle p}\downarrow & \rightsquigarrow & \downarrow{\scriptstyle q(p)} \\ \mathcal{B} & & \mathcal{Q}(p) \end{array}$$

Objects in $\mathcal{Q}(p)$ are p-equivalence relations, that is, pairs of $I \in \mathcal{B}$ and $\rho \in \mathcal{E}_{I\times I}$ s.t.

$$\begin{cases} \bullet\ \delta_I \leq \rho \text{ in } \mathcal{E}_{I\times I} \text{ where } \delta_I = \exists_{\Delta_I}(\top) \\ \bullet\ \rho \leq \rho[\langle \pi_2, \pi_1\rangle] \text{ in } \mathcal{E}_{I\times I} \\ \bullet\ \rho[\langle \pi_1, \pi_2\rangle] \wedge \rho[\langle \pi_2, \pi_3\rangle] \leq \rho[\langle \pi_1, \pi_3\rangle] \text{ in } \mathcal{E}_{I\times I\times I} \end{cases}$$

Function comprehension (Pavlović. '96)

p is function-comprehensive

if $\boxed{\begin{array}{l} \text{for any } \rho \in \mathcal{E}_{I\times J} \text{ with} \\ \begin{cases} \bullet\ \rho[\langle \pi_1, \pi_2\rangle] \wedge \rho[\langle \pi_1, \pi_3\rangle] \leq \delta[\langle \pi_2, \pi_3\rangle] \\ \bullet\ \top \leq \exists_{\pi_1}(\rho), \end{cases} \\ \text{there is } u : I \to J \text{ in } \mathcal{B} \text{ s.t. } \rho \cong \delta[u \times id_J]. \end{array}}$

= "Single-valued total relations come from a function."

The paper mentions function comprehensive completion.

Motivation

cumbersome constructions

$$\begin{array}{ccc} \boxed{\text{fibrations with } ☆} & \longrightarrow & \boxed{\text{fibrations with } ♡} \\ \downarrow & & \uparrow \\ \boxed{\text{relational structures with } ☆'} & \xrightarrow{\text{clearer?}} & \boxed{\text{relational structures with } ♡'} \end{array}$$

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Virtual double categories of relations

The double category $\mathbb{R}\mathrm{el}$:

objects ⋯ sets

tight arrows $\begin{array}{c} I \\ \downarrow u \\ J \end{array}$ ⋯ functions

loose arrows $I \overset{\alpha}{\nrightarrow} K$ ⋯ binary relations

cells
$$\begin{tikzcd} I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\subseteq"] & K \arrow[d, "v"] \\ J \arrow[r, "\beta"', "\shortmid"{marking}] & L \end{tikzcd}$$
⋯ (proofs of) implication $\alpha(x,y) \Rightarrow \beta(u(x), v(y))$

The loosewise composition is given by

$I \overset{Id_I}{\nrightarrow} I$ ⋯ $\{ (x, x') \mid x = x' \}$

$I \overset{\alpha}{\nrightarrow} J \overset{\beta}{\nrightarrow} K$ ⋯ $\left\{ (x, z) \;\middle|\; \exists y \in J.\ \begin{array}{l} \alpha(x,y) \\ \wedge\ \beta(y,z) \end{array} \right\}$

$$\begin{tikzcd} \boxed{\wedge,\ =,\ \exists} \arrow[d, no head] \arrow[r, dotted, no head] & \boxed{\text{Double categories}} \arrow[d, no head] \\ \boxed{\wedge,\ =} \arrow[d, no head] \arrow[r, dotted, no head] & \boxed{?} \arrow[d, no head] \\ \boxed{\wedge} \arrow[r, dotted, no head] & \boxed{?} \end{tikzcd}$$

Def A virtual double category $\mathbb{D}$ consists of

- objects $I, J, \dots$
- tight arrows $\begin{array}{c} I \\ \downarrow u \\ J \end{array}, \dots$
- loose arrows $I \overset{\alpha}{\nrightarrow} K, \dots$
- (virtual) cells
$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "u"'] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "v"] \\ J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] \arrow[rrru, phantom, "\nu"] & & & J_1 \end{tikzcd}$$
for $n \geq 0$

with compositions of tight arrows and cells, satisfying some laws.

Virtual double categories of relations

Example (A DC is a VDC)

In $\mathbb{R}\mathrm{el}$, a virtual cell

$$\begin{tikzcd} I_0 \arrow[r, "\alpha", "\shortmid" marking] \arrow[d, "u"'] & I_1 \arrow[r, "\beta", "\shortmid" marking] \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\subseteq$}"] & I_2 \arrow[d, "v"] \\ J_0 \arrow[rr, "\gamma"', "\shortmid" marking] & & J_2 \end{tikzcd}$$

exists if and only if

$$\left(\exists y \in I_2.\ \alpha(x,y) \wedge \beta(y,z)\right) \Rightarrow \gamma(u(x), v(z)),$$
$$(\forall x, z)$$

but this is equivalent to

$$\alpha(x,y) \wedge \beta(y,z) \Rightarrow \gamma(u(x), v(z)) \quad (\forall x, y, z).$$

By adopting this presentation, weaker systems than regular logic are modeled in a VDC with "substitution" and "finite products".

Def A restriction of
$$\begin{tikzcd} I \arrow[d, "u"'] & K \arrow[d, "v"] \\ J \arrow[r, "\alpha"', "\shortmid" marking] & L \end{tikzcd}$$
is a loose arrow $I \xrightarrow{\alpha[u;v]} K$, equipped with a cell

$$\begin{tikzcd} I \arrow[r, "{\alpha[u;v]}", "\shortmid" marking] \arrow[d, "u"'] \arrow[dr, phantom, "\rho"] & K \arrow[d, "v"] \\ J \arrow[r, "\alpha"', "\shortmid" marking] & L \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} M_0 \arrow[r, "\shortmid" marking] \arrow[d, "s"'] & \cdots \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\forall\ \nu"] & M_n \arrow[d, "t"] \\ I \arrow[d, "u"'] & & K \arrow[d, "v"] \\ J \arrow[rr, "\alpha"', "\shortmid" marking] & & L \end{tikzcd} = \begin{tikzcd} M_0 \arrow[r, "\shortmid" marking] \arrow[d, "s"'] & \cdots \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\exists!\ \tilde{\nu}"] & M_n \arrow[d, "t"] \\ I \arrow[rr, "\shortmid" marking] \arrow[d, "u"'] & \arrow[d, phantom, "\rho"] & K \arrow[d, "v"] \\ J \arrow[rr, "\alpha"', "\shortmid" marking] & {} & L \end{tikzcd}$$

Def A VDC $\mathbb{D}$ is cartesian fibrational if

- every $\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "\shortmid" marking] & \bullet \end{tikzcd}$ in $\mathbb{D}$ has a restriction,
- the tight category $\mathbb{D}_t$ has finite products,
- the loose hom-categories $\mathbb{D}(I, J)$ have finite products preserved by $(-)[u;v]$'s.

Def A *restriction* of
$$\begin{tikzcd} I \arrow[d, "u"'] & K \arrow[d, "v"] \\ J \arrow[r, "\alpha"', "\shortmid" marking] & L \end{tikzcd}$$
is a loose arrow $I \xrightarrow{\alpha[u;v]} K$, equipped with a cell
$$\begin{tikzcd} I \arrow[r, "{\alpha[u;v]}", "\shortmid" marking] \arrow[d, "u"'] \arrow[dr, phantom, "\rho"] & K \arrow[d, "v"] \\ J \arrow[r, "\alpha"', "\shortmid" marking] & L \end{tikzcd}$$
with the following universal property:
$$\begin{tikzcd} M_0 \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & M_n \arrow[d, "t"] \\ I \arrow[u, leftarrow, "s"] \arrow[d, "u"'] \arrow[urr, phantom, "\forall"] \arrow[rr, phantom, "\nu"] & & K \arrow[d, "v"] \\ J \arrow[rr, "\alpha"', "\shortmid" marking] & & L \end{tikzcd}
=
\begin{tikzcd} M_0 \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & M_n \arrow[d, "t"] \\ I \arrow[u, leftarrow, "s"] \arrow[rr, "\shortmid" marking] \arrow[d, "u"'] \arrow[urr, phantom, "\exists!\ \tilde{\nu}"] \arrow[drr, phantom, "\rho"] & & K \arrow[d, "v"] \\ J \arrow[rr, "\alpha"', "\shortmid" marking] & & L \end{tikzcd}$$

Def A VDC $\mathbb{D}$ is *cartesian fibrational* if

- every $\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, "\shortmid" marking] & \bullet \end{tikzcd}$ in $\mathbb{D}$ has *a restriction*,
- the tight category $\mathbb{D}_t$ has finite products,
- the loose hom-categories $\mathbb{D}(I, J)$ have finite products *preserved by* $(-)[u;v]$'s.

Now, we construct the VDC of *relations relative to* $\mathbb{P}$.

Def (Shulman.'08 $\mathbb{F}\mathrm{r}$, Lawler.'15 $\mathbb{M}\mathrm{atr}$, Nasu.'25)

A VDC $\mathbb{B}\mathrm{il}(\mathbb{P})$ for a cartesian fibration $\begin{tikzcd} \mathcal{E} \arrow[d, "\mathbb{P}"] \\ \mathcal{B} \end{tikzcd}$ is defined as follows:

- its tight category $\mathbb{B}\mathrm{il}(\mathbb{P})_0$: $\mathcal{B}$
- its loose arrows $I \xrightarrow{\alpha} J$: $\alpha \in \mathcal{E}_{I \times J}$
- cells $\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid" marking] \arrow[d, "u"'] & \cdots \arrow[r, "\alpha_n", "\shortmid" marking] \arrow[d, phantom, "\nu"] & I_n \arrow[d, "v"] \\ J_0 \arrow[rr, "\beta"', "\shortmid" marking] & & J_1 \end{tikzcd}$:

$$\alpha_1[\langle \pi_0, \pi_1 \rangle] \wedge \cdots \wedge \alpha_n[\langle \pi_{n-1}, \pi_n \rangle] \longrightarrow \beta[(u \times v) \circ \langle pr_0, pr_n \rangle]$$
in $\mathcal{E}_{I_0 \times \cdots \times I_n}$

proofs of $\alpha_1(x_0, x_1) \wedge \cdots \wedge \alpha_n(x_{n-1}, x_n)$
$\Longrightarrow \beta(u(x_0), v(x_n))$

Prop $\mathbb{B}\mathrm{il}(\mathbb{P})$ is a cartesian fibrational VDC.

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Comparing fibrations and VDCs

$\wedge, =, \exists$
|
$\wedge, =$
|
$\wedge$

elementary existential fibrations
|
elementary fibrations
|
cartesian fibrations $\xrightarrow[\mathbb{B}\mathrm{il}]{}$ cartesian fibrational VDC

cartesian equipments
|
cartesian virtual equipments
|
cartesian fibrational VDC

Equipments and virtual equipments can be / are formulated as VDCs with a certain kind of composition in terms of universal properties.

Def A composite of $I_0 \overset{\alpha_1}{\nrightarrow} \cdots \overset{\alpha_n}{\nrightarrow} I_n$ is a loose arrow $I_0 \overset{\odot\alpha}{\nrightarrow} I_n$ together with

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] \arrow[d, phantom, "\varpi"] & I_n \arrow[d, equal] \\
I_0 \arrow[rr, "\odot\alpha"', "\shortmid"{marking}] & & I_n
\end{tikzcd}$$

with the following universal property :

$$\begin{tikzcd}
J \arrow[r, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "u"'] & I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] \arrow[d, phantom, "\forall\ \nu"] & I_n \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] & K \arrow[d, "v"] \\
L \arrow[rrrr, "l"', "\shortmid"{marking}] & & & & M
\end{tikzcd}
=
\begin{tikzcd}
J \arrow[r, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, equal] & I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] \arrow[d, phantom, "\varpi"] & I_n \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] & K \arrow[d, equal] \\
J \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "u"'] & I_0 \arrow[rr, "\shortmid"{marking}] & \arrow[d, phantom, "\exists!\ \widetilde{\nu}"] & I_n \arrow[r, dashed, "\shortmid"{marking}] & K \arrow[d] \\
L \arrow[rrrr, "l"', "\shortmid"{marking}] & & & & M
\end{tikzcd}$$

Comparing fibrations and VDCs

$\wedge, =, \exists$	elementary existential fibrations	$\xrightarrow{\mathbb{B}\mathrm{il}}$	cartesian equipments
$\wedge, =$	elementary fibrations	$\longrightarrow$	cartesian virtual equipments
$\wedge$	cartesian fibrations	$\xrightarrow[\mathbb{B}\mathrm{il}]{}$	cartesian fibrational VDC

$\boxed{\text{Prop}}$

When $\mathbb{P}$ is $\left\{\begin{array}{c}\text{elementary}\\ \text{elementary existential}\end{array}\right\}$, $\mathbb{B}\mathrm{il}(\mathbb{P})$ is $\left\{\begin{array}{l}\text{a cartesian virtual equipment.}\\ \text{a cartesian equipment.}\end{array}\right.$

More precisely, we have 2-functors

$$\begin{tikzcd}
\mathbf{EEFib} \arrow[r] \arrow[d] & \mathbf{CartEq}' \arrow[d] \\
\mathbf{EleFib} \arrow[r] \arrow[d] & \mathbf{CartVEq} \arrow[d] \\
\mathbf{CartFib} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \mathbf{CartFVDC}
\end{tikzcd}.$$

Comparing fibrations and VDCs

$$\begin{tikzcd}
\boxed{\wedge\,,\,=\,,\,\exists} \arrow[d, dash] & \boxed{\text{elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d, dash] & \boxed{\text{cartesian equipments}} \arrow[d, dash] \\
\boxed{\wedge\,,\,=} \arrow[d, dash] & \boxed{\text{elementary fibrations}} \arrow[r] \arrow[d, dash] & \boxed{\text{cartesian virtual equipments}} \arrow[d, dash] \\
\boxed{\wedge} & \boxed{\text{cartesian fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \boxed{\text{cartesian fibrational VDC}}
\end{tikzcd}$$

Thm For a cartesian fibration P,

P is elementary existential $\iff$ $\mathbb{B}\mathrm{il}(P)$ is a cartesian equipment

Moreover, we have a 2-pullback

$$\begin{tikzcd}
\mathrm{EEFib} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{CartEq}' \arrow[d] \\
\mathrm{CartFib} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \mathrm{CartFVDC}
\end{tikzcd}$$

Prop $\mathbb{B}\mathrm{il} : \mathrm{EEFib} \to \mathrm{CartEq}$ is locally an equivalence.

The essential image of this is characterized by Frobenius axiom.
(Walters, Wood '08
Nasu '25 2.3.2)

Comparing fibrations and VDCs

$\boxed{\text{Example}}$

From the fibration $\begin{matrix}\mathcal{M}\\ \downarrow \\ \mathcal{C}\end{matrix}$ for a stable factorization system (E, M), we obtain

$$\mathbb{B}\mathrm{il}\left(\begin{matrix}\mathcal{M}\\ \downarrow \\ \mathcal{C}\end{matrix}\right) = \underline{\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})}$$

the double category of relations relative to (E, M). (Hoshino, Nasu. '23)

$\boxed{\text{Application}}$ (Function comprehension)

For an elementary existential fibration $p : \mathcal{E} \to \mathcal{B}$, a total and single-valued relation $\lambda \in \mathcal{E}_{I \times J}$ is equivalent to a loose adjunction

$$\begin{tikzcd} I \arrow[r, shift left=1.5, "\lambda", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & J \arrow[l, shift left=1.5, "\rho", "\shortmid"{marking}] \end{tikzcd} \quad \text{in } \mathbb{B}\mathrm{il}(p).$$

The function comprehension property is translated as Cauchyness of double categories:

$$\forall \; \begin{tikzcd} I \arrow[r, shift left=1.5, "\lambda", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & J \arrow[l, shift left=1.5, "\rho", "\shortmid"{marking}] \end{tikzcd} \quad \exists \; \begin{tikzcd} I \arrow[d, "u"'] \\ J \end{tikzcd} \quad (\lambda \dashv \rho) \text{“}\cong\text{”} (u_* \dashv u^*)$$

The function comprehensive completion is achieved as follows:

$$\begin{tikzcd}[column sep=large] \mathbf{EEFib} \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] \arrow[d, "\mathbb{B}\mathrm{il}"'] & \mathbf{EEFib}_{\mathrm{funccomp}} \arrow[l, shift left=1.5] \\ \mathbf{CartEq} \arrow[r, shift left=1.5, "\mathbb{C}\mathrm{au}"] \arrow[r, phantom, "\perp"] & \mathbf{CartEq}_{\mathrm{Cauchy}} \arrow[l, shift left=1.5] \arrow[u] \end{tikzcd}$$

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Conclusion

- introduced cartesian fibrational virtual double categories as a relation-oriented framework for predicate logic.
- presented the $\mathbb{B}\mathrm{il}$-construction which produces a cartesian fibrational VDC from a cartesian fibration.
- $$\begin{tikzcd} \mathbf{EEFib} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{CartEq}' \arrow[d] \\ \mathbf{CartFib} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \mathbf{CartFVDC} \end{tikzcd}$$

Others in my thesis

- Recovering a characterization theorem in (Hoshino, Nasu. '23).
- Connection to bicategories.
- Self-duality on Frobenius equipments.

(• An internal logic of fibrational VDCs.)

Future Work

- develop theories of exact completion and other kinds of completion in terms of virtual double categories.
- ?

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Thank you!

my homepage :

hayatonasu.github.io

my thesis :

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