

From Fibrations

to Virtual Double Categories

Categorical Logic Meets Double Categories – Side A

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Outline

01 / 20

This talk is based on my recent thesis:

Logical Aspects of
Chapter 2 : Virtual Double Categories
Categorical Logic meets
Double Categories

Today's slides



Fibrations (Lawvere.'70, Jacobs.'99)

- ✓ terms (functions)
- ✗ binary relations

Virtual Double Categories

- ✓ terms (functions)
- ✓ binary relations

Bicategories (Carboni, Walters.'87
Freyd, Scedrov.'90)

- ✗ terms (functions)
- ✓ binary relations

Today's
focus

1. Fibrations and Logic
2. Virtual Double Categories
of Relations
3. Comparing Fibrations
and Virtual Double Categories
4. Conclusion

1. Fibrations and Logic

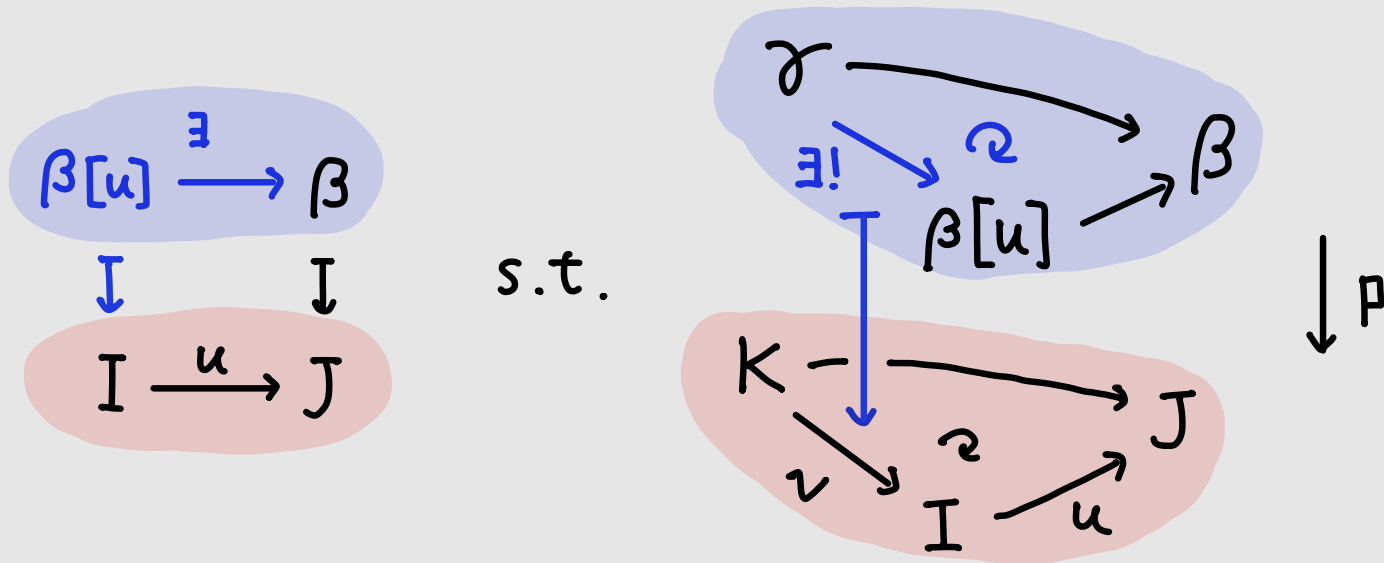
2. Virtual Double Categories
of Relations

3. Comparing Fibrations
and Virtual Double Categories

4. Conclusion

Taking semantics in fibrations

A functor $\mathcal{E} \downarrow^{\mathcal{P}} \mathcal{B}$ is a (Grothendieck) fibration



Example

$(\mathcal{E}, \mathcal{M})$: a stable factorization system on \mathcal{C}

$\rightsquigarrow \mathcal{M} \downarrow \mathcal{C}$ (\mathcal{M} : the full subcategory of $\mathcal{C}^{\rightarrow}$ spanned by $m \in \mathcal{M}$) is a fibration.

Idea

\mathcal{B} ... the category of contexts Γ

and sequences of terms $\Gamma \vdash S : \Delta$

$$\begin{array}{c} x_1 : R, x_2 : R, x_3 : M \\ \downarrow (x_2 \cdot x_3, x_1 - x_2) \\ y_1 : M, y_2 : R \end{array}$$

\mathcal{E}_{Γ} (the fiber category)

... the category of formulas $\Gamma \vdash \varphi$

and proofs of implication $\Gamma \vdash p : \varphi \Rightarrow \psi$

$$\begin{array}{c} \exists y : R (y + x = y) \\ \downarrow \\ x = 0 \end{array} \quad \mathcal{E}_{x : R}$$

Taking semantics in fibrations

Given a language \mathcal{L} , what structures do we need on a fibration to interpret formulas in \mathcal{L} ?

- a sort $\sigma \in \mathcal{L} \rightsquigarrow$ an object $[[\sigma]]$ in \mathcal{B}

- a context $x_1 : \sigma_1, \dots, x_n : \sigma_n$

\rightsquigarrow the **product** $[[\sigma_1]] \times \dots \times [[\sigma_n]]$

- a function symbol $f : \sigma_1, \dots, \sigma_n \rightarrow \tau$

\rightsquigarrow an arrow $[[f]] : [[\sigma_1]] \times \dots \times [[\sigma_n]] \rightarrow [[\tau]]$ in \mathcal{B}

- the interpretation of terms is inductively defined

e.g. $[[f(g(x), y)]] = [[f]] \circ ([[g]] \times \text{id})$

- a relation symbol $R : \sigma_1, \dots, \sigma_n$

\rightsquigarrow an object $[[R]]$ in the **fiber** of $[[\sigma_1]] \times \dots \times [[\sigma_n]]$

- substitution $\mathcal{V} [t_1/x_1, \dots, t_n/x_n]$

$$[[\mathcal{V} [t_1/x_1, \dots, t_n/x_n]]] \longrightarrow [[\mathcal{V}]]$$

\rightsquigarrow

$$\downarrow \qquad \langle [[t_1]], \dots, [[t_n]] \rangle \qquad \downarrow$$

$$[[\tau_1]] \times \dots \times [[\tau_n]] \longrightarrow [[\sigma_1]] \times \dots \times [[\sigma_n]]$$

using the cartesian (prone) lifting.

- the conjunction $\mathcal{V} \wedge \mathcal{V}' \rightsquigarrow$ **fiberwise product**
 $[[\mathcal{V}]] \wedge [[\mathcal{V}']]$

- the existential quantifier $\exists z : \sigma. \mathcal{V}(\vec{x}, z)$

\rightsquigarrow the image of $[[\mathcal{V}(\vec{x}, z)]] \in \Sigma_{[[\Gamma]] \times [[\sigma]]}$

by the left adjoint

$$\Sigma_{[[\Gamma]] \times [[\sigma]]} \begin{array}{c} \xrightarrow{\exists \pi} \\ \perp \\ \xleftarrow{(-) [\pi]} \end{array} \Sigma_{[[\Gamma]]}$$

($\pi : [[\Gamma]] \times [[\sigma]] \rightarrow [[\Gamma]]$)

Taking semantics in fibrations

A sufficient structure for regular logic is an **elementary existential doctrine** by Lawvere.

Def A fibration \mathcal{P} is **cartesian** if

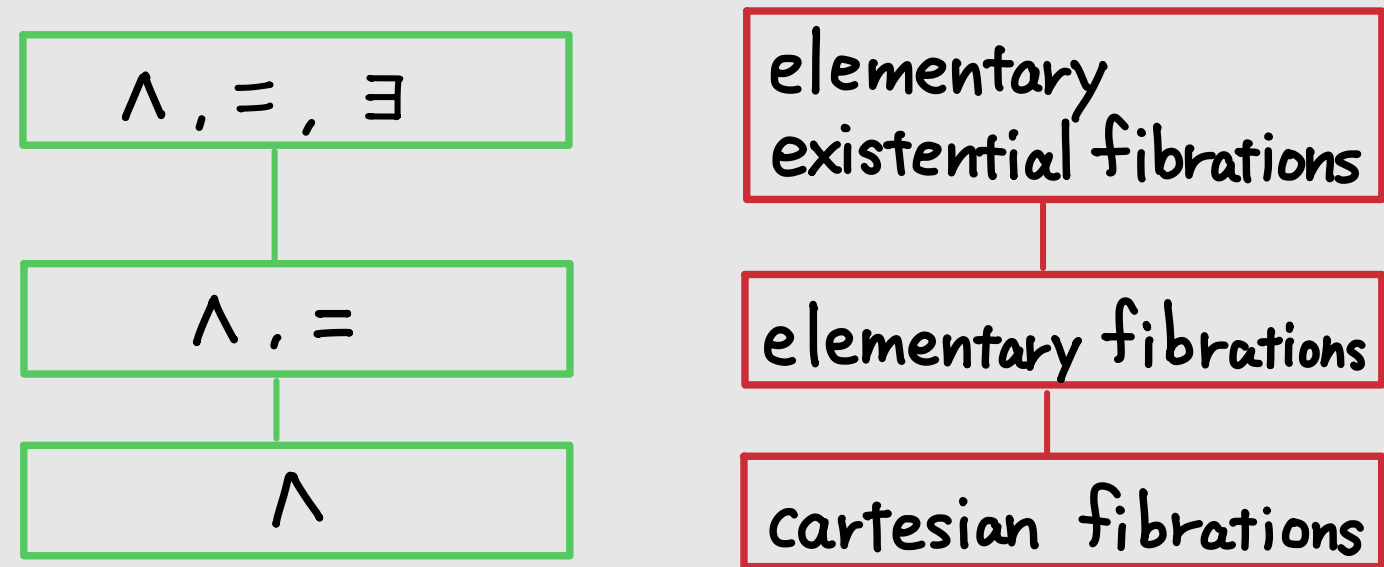
- the base category \mathcal{B} has finite products,
- the fiber categories \mathcal{E}_I have finite products, and the reindexings $(-)[u]$ preserve them.

Def A fibration \mathcal{P} is **elementary existential** if

- it is cartesian,
- all reindexing functors $(-)[u]$ for $u \rightarrow$ of the forms $\begin{cases} I \times J \xrightarrow{\pi_2} J \\ I \times J \xrightarrow{\Delta \times id} I \times I \times J \end{cases}$ have a **left adjoint** $\exists u$, and

- it satisfies some axioms.
(the Beck-Chevalley condition)
(the Frobenius reciprocity)

We have hierarchies of logical systems and classes of fibrations.



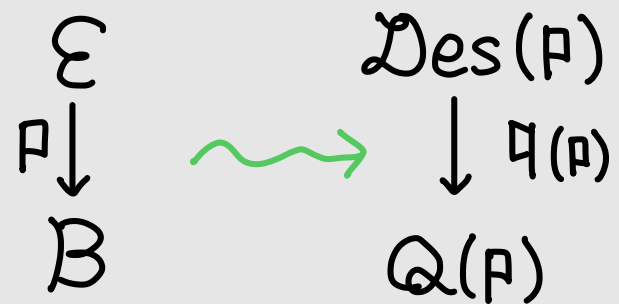
We will add a new column of hierarchy: the hierarchy of virtual double categories.

Constructing a new fibration

⚠ Fibers are pre-ordered in this slide.

Quotient completion (Maietti, Rosolini, '13)

The quotient completion is an analogue of the exact completion in the context of doctrines.



Objects in $\mathcal{Q}(\mathcal{P})$ are \mathcal{P} -equivalence relations, that is, pairs of $I \in \mathcal{B}$ and $\rho \in \mathcal{E}_{I \times I}$ s.t.

- $\delta_I \leq \rho$ in $\mathcal{E}_{I \times I}$ where $\delta_I = \exists_{\Delta_I}(\tau)$
- $\rho \leq \rho[\langle \pi_2, \pi_1 \rangle]$ in $\mathcal{E}_{I \times I}$
- $\rho[\langle \pi_1, \pi_2 \rangle] \wedge \rho[\langle \pi_2, \pi_3 \rangle] \leq \rho[\langle \pi_1, \pi_3 \rangle]$ in $\mathcal{E}_{I \times I * I}$

Function comprehension (Pavlović, '96)

\mathcal{P} is function-comprehensive

if for any $\rho \in \mathcal{E}_{I \times J}$ with

- $\rho[\langle \pi_1, \pi_2 \rangle] \wedge \rho[\langle \pi_1, \pi_3 \rangle] \leq \delta[\langle \pi_2, \pi_3 \rangle]$
- $\tau \leq \exists_{\pi_1}(\rho)$,

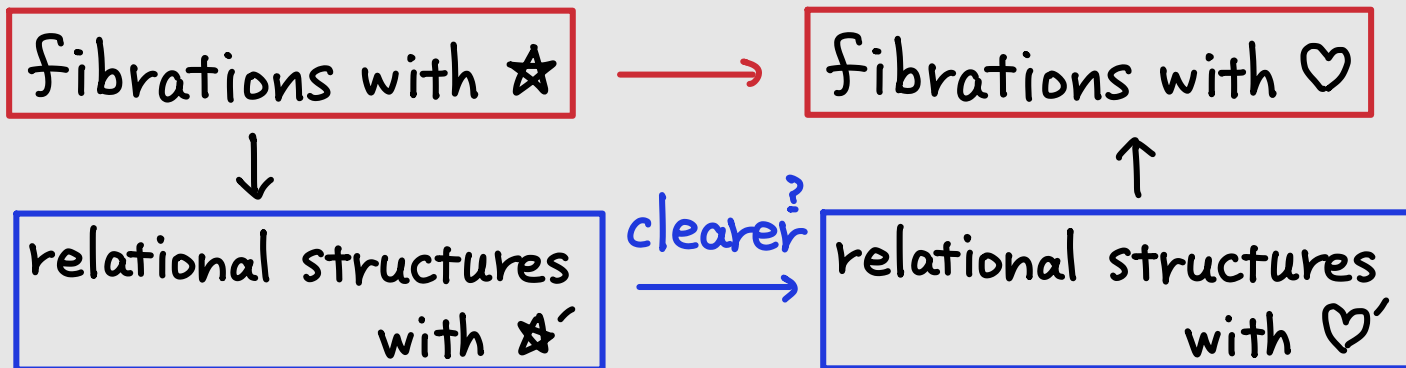
there is $u: I \rightarrow J$ in \mathcal{B} s.t. $\rho \cong \delta[u \times \text{id}_J]$.

= "Single-valued total relations come from a function."

The paper mentions function comprehensive completion.

Motivation

cumbersome constructions



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Virtual double categories of relations

The double category $\mathbb{R}el$:

objects ... sets

tight arrows $I \downarrow u \dots J$... functions

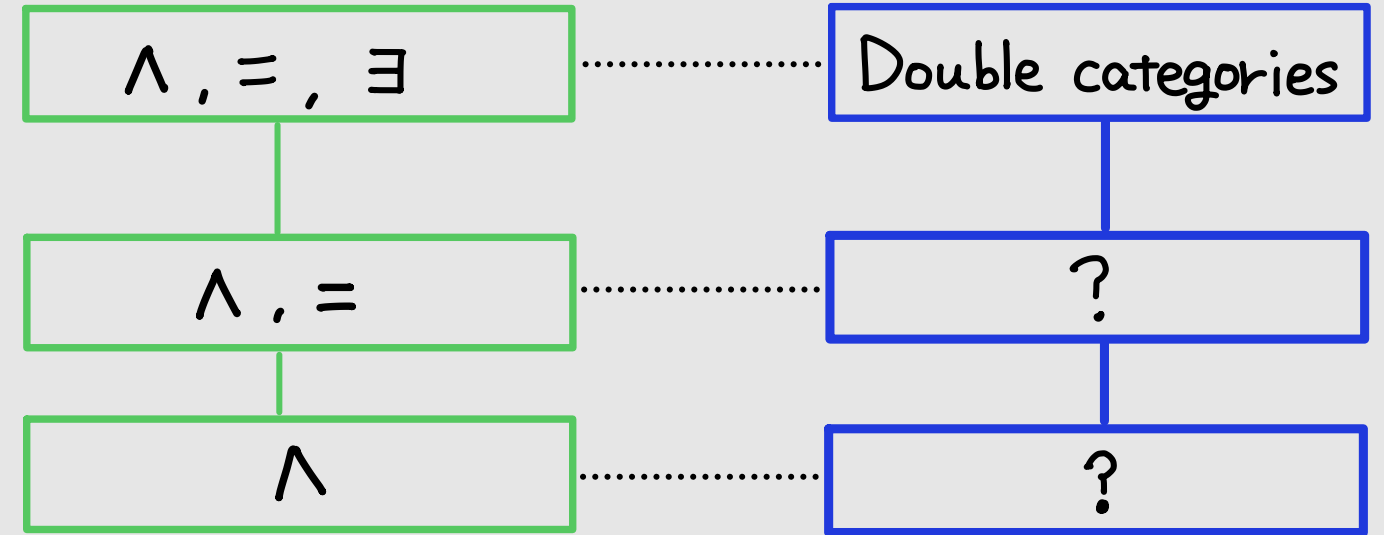
loose arrows $I \xrightarrow{\alpha} K$... binary relations

cells $I \xrightarrow{\alpha} K$
 $u \downarrow \quad \downarrow v$
 $J \xrightarrow{\beta} L$... (proofs of) implication
 $\alpha(x, y) \Rightarrow \beta(u(x), v(y))$

The loosewise composition is given by

$$I \xrightarrow{Id_I} I \dots \{ (x, x') \mid x = x' \}$$

$$I \xrightarrow{\alpha} J \xrightarrow{\beta} K \dots \left\{ (x, z) \mid \exists y \in J. \begin{matrix} \alpha(x, y) \\ \wedge \beta(y, z) \end{matrix} \right\}$$



Def A virtual double category \mathbb{D} consists of

- objects I, J, \dots
- tight arrows $I \downarrow u \dots J$

- loose arrows $I \xrightarrow{\alpha} K, \dots$

- (virtual) cells $I_0 \xrightarrow{\alpha_1} I_1 \xrightarrow{\dots} I_n$
 $u \downarrow \quad \downarrow v$
 $J_0 \xrightarrow{\beta} J_1$ for $n \geq 0$

with compositions of tight arrows and cells, satisfying some laws.

Virtual double categories of relations

Example (A DC is a VDC)

In Rel, a virtual cell

$$\begin{array}{ccccc} I_0 & \xrightarrow{\alpha} & I_1 & \xrightarrow{\beta} & I_2 \\ u \downarrow & & \cap_1 & & \downarrow v \\ J_0 & \xrightarrow{\gamma} & & & J_2 \end{array}$$

exists if and only if

$$(\exists y \in I_2. \alpha(x, y) \wedge \beta(y, z)) \Rightarrow \gamma(u(x), v(z)),$$

(∀ x, z)

but this is equivalent to

$$\alpha(x, y) \wedge \beta(y, z) \Rightarrow \gamma(u(x), v(z)) \quad (\forall x, y, z).$$

By adopting this presentation, weaker systems than regular logic are modeled in a VDC with "substitution" and "finite products".

Def

A restriction of $u \downarrow \begin{array}{c} I \\ J \end{array} \xrightarrow{\alpha} L$ $\downarrow v$ $\begin{array}{c} K \\ L \end{array}$

is a loose arrow $I \xrightarrow{\alpha[u;v]} K$, equipped with a cell

$$\begin{array}{ccc} I & \xrightarrow{\alpha[u;v]} & K \\ u \downarrow & \rho & \downarrow v \\ J & \xrightarrow{\alpha} & L \end{array}$$

with the following universal property:

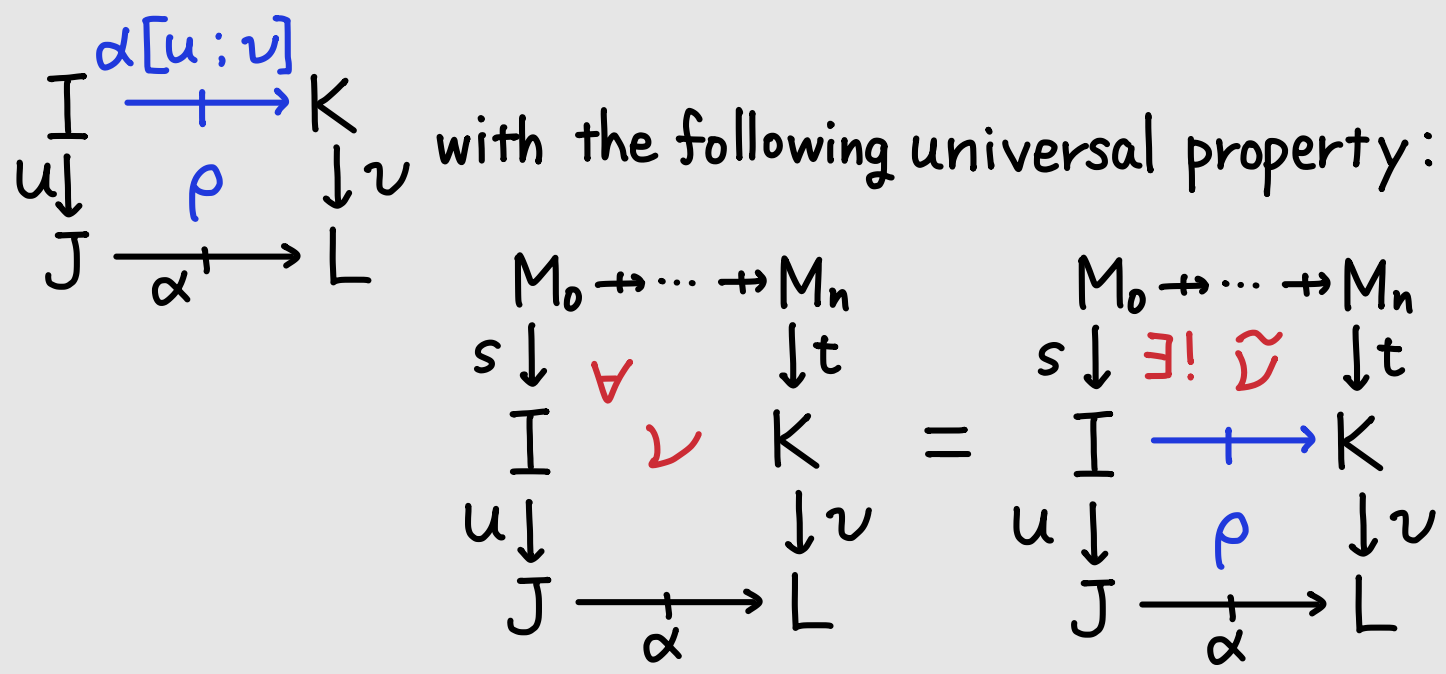
$$\begin{array}{ccc} M_0 \rightarrow \dots \rightarrow M_n & & M_0 \rightarrow \dots \rightarrow M_n \\ s \downarrow & \forall & \downarrow t \\ I & \xrightarrow{\gamma} & K \\ u \downarrow & & \downarrow v \\ J & \xrightarrow{\alpha} & L \end{array} = \begin{array}{ccc} I & \xrightarrow{\alpha[u;v]} & K \\ u \downarrow & \rho & \downarrow v \\ J & \xrightarrow{\alpha} & L \end{array}$$

Def

A VDC \mathbb{D} is cartesian fibrational if

- every $\begin{array}{c} \downarrow \\ \cdot \end{array} \rightarrow \begin{array}{c} \downarrow \\ \cdot \end{array}$ in \mathbb{D} has a restriction,
- the tight category \mathbb{D}_t has finite products,
- the loose hom-categories $\mathbb{D}(I, J)$ have finite products preserved by $(-)[u;v]$'s.

Def A restriction of $\begin{array}{ccc} I & & K \\ u \downarrow & & \downarrow v \\ J & \xrightarrow{\alpha} & L \end{array}$ is a loose arrow $I \xrightarrow{\alpha[u;v]} K$, equipped with a cell



Def A VDC \mathbb{D} is cartesian fibrational if

- every $\begin{array}{ccc} \cdot & \twoheadrightarrow & \cdot \\ \downarrow & & \downarrow \end{array}$ in \mathbb{D} has a restriction,
- the tight category \mathbb{D}_t has finite products,
- the loose hom-categories $\mathbb{D}(I, J)$ have finite products preserved by $(-)[u;v]$'s.

Now, we construct the VDC of relations relative to \mathcal{P} .

Def (Shulman '08 Fr, Lawler '15 Matr, Nasu '25)

A VDC $\mathbb{Bil}(\mathcal{P})$ for a cartesian fibration $\begin{array}{c} \mathcal{E} \\ \downarrow \mathcal{P} \\ \mathcal{B} \end{array}$ is defined as follows:

- its tight category $\mathbb{Bil}(\mathcal{P})_0 : \mathcal{B}$
- its loose arrows $I \xrightarrow{\alpha} J : \alpha \in \mathcal{E}_{I \times J}$
- cells $\begin{array}{ccc} I_0 & \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} & I_n \\ u \downarrow & \nu & \downarrow v \\ J_0 & \xrightarrow{\beta} & J_1 \end{array} :$

$$\alpha_1[\langle \pi_0, \pi_1 \rangle] \wedge \dots \wedge \alpha_n[\langle \pi_{n-1}, \pi_n \rangle] \longrightarrow \beta[(u \times v) \circ \langle \text{pr}_0, \text{pr}_n \rangle]$$

in $\mathcal{E}_{I_0 \times \dots \times I_n}$

proofs of $\alpha_1(x_0, x_1) \wedge \dots \wedge \alpha_n(x_{n-1}, x_n) \implies \beta(u(x_0), v(x_n))$

Prop $\mathbb{Bil}(\mathcal{P})$ is a cartesian fibrational VDC.

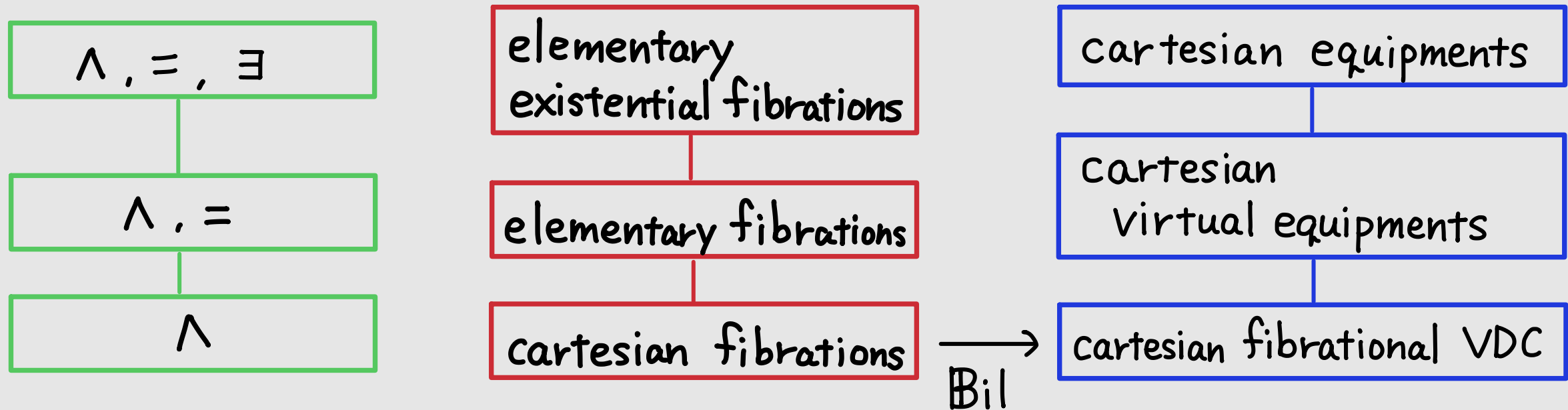
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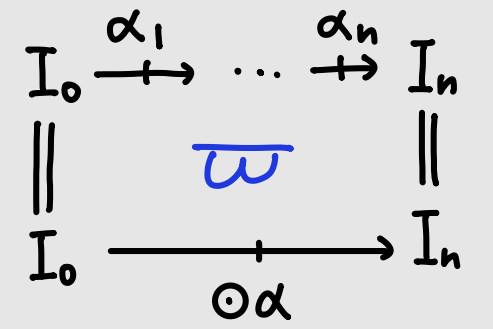
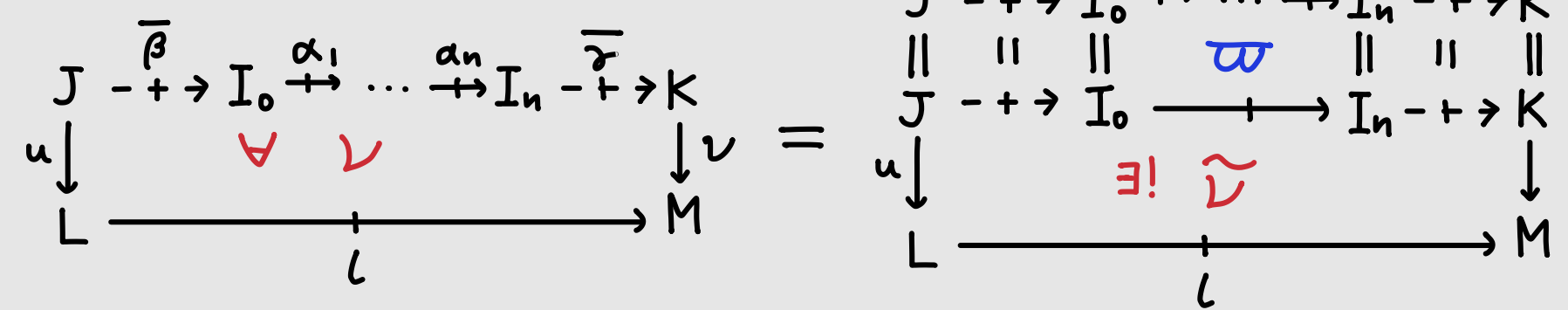
Comparing fibrations and VDCs



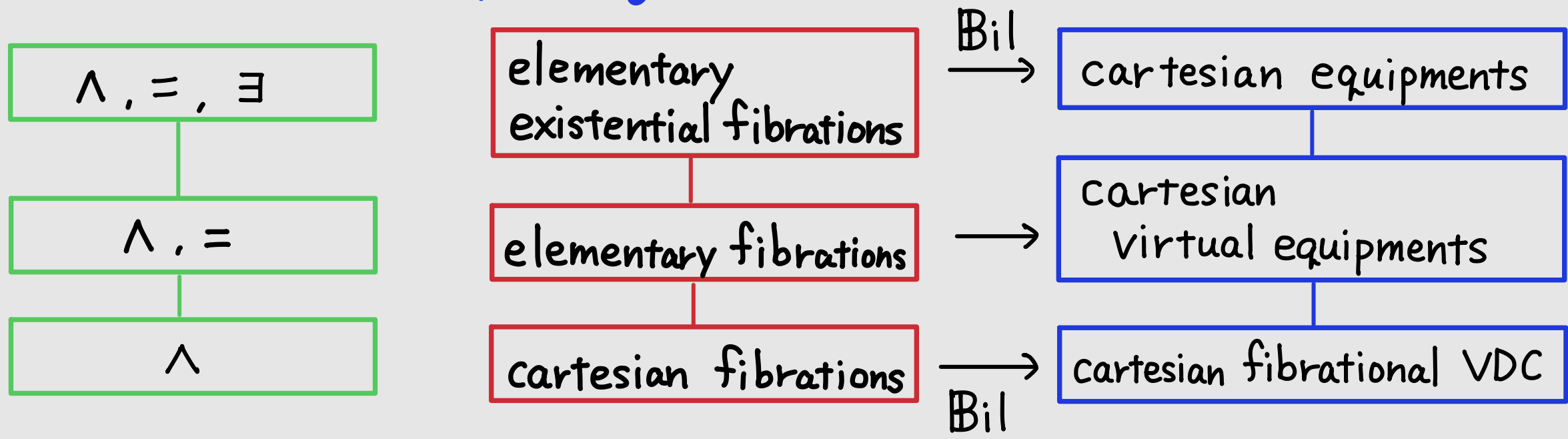
Equipments and virtual equipments can be / are formulated as VDCs with a certain kind of composition in terms of universal properties.

Def A composite of $I_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} I_n$ is a loose arrow $I_0 \xrightarrow{\odot \alpha} I_n$ together with

with the following universal property :



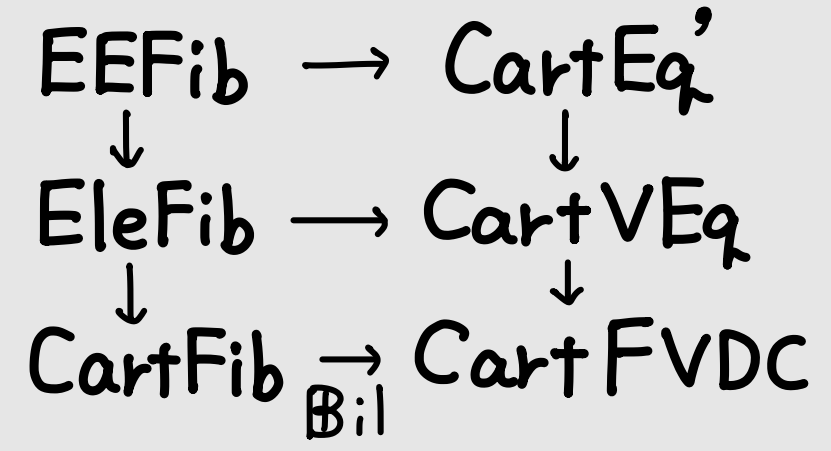
Comparing fibrations and VDCs



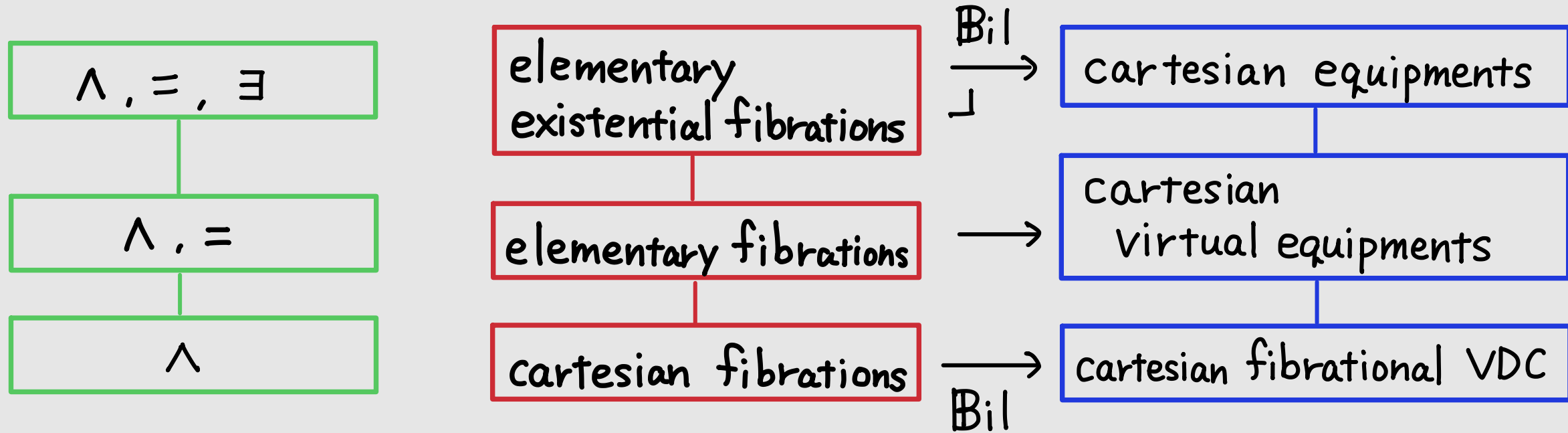
Prop

When \mathcal{F} is $\left\{ \begin{array}{l} \text{elementary} \\ \text{elementary existential} \end{array} \right\}$, $\text{Bil}(\mathcal{F})$ is $\left\{ \begin{array}{l} \text{a cartesian virtual equipment.} \\ \text{a cartesian equipment.} \end{array} \right.$

More precisely, we have 2-functors



Comparing fibrations and VDCs



Thm For a cartesian fibration \mathbb{P} ,

\mathbb{P} is elementary existential $\iff \mathbb{B}il(\mathbb{P})$ is a cartesian equipment

Moreover, we have a 2-pullback

$$\begin{array}{ccc}
 \mathbb{E}EFib & \xrightarrow{\quad} & \mathbb{C}artEq' \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbb{C}artFib & \xrightarrow[\mathbb{B}il]{} & \mathbb{C}artFVDC
 \end{array}$$

Prop $\mathbb{B}il : \mathbb{E}EFib \rightarrow \mathbb{C}artEq$ is locally an equivalence.

The essential image of this is characterized by Frobenius axiom.
 (Walters, Wood '08)
 (Nasu'25 2.3.2)

Comparing fibrations and VDCs

Example

From the fibration $\mathcal{M} \downarrow_{\mathcal{E}}$ for a stable factorization system (E, M) , we obtain

$$\text{Bil} \left(\mathcal{M} \downarrow_{\mathcal{E}} \right) = \text{Rel}_{(E, M)}(\mathcal{E})$$

the double category of relations relative to (E, M) . (Hoshino, Nasu, '23)

Application (Function comprehension)

For an elementary existential fibration $\beta: \mathcal{E} \rightarrow \mathcal{B}$, a total and single-valued relation $\lambda \in \mathcal{E}_{I \times J}$

is equivalent to a loose adjunction

$$I \begin{array}{c} \xrightarrow{\lambda} \\ \perp \\ \xleftarrow{\rho} \end{array} J \quad \text{in } \text{Bil}(\beta).$$

The function comprehension property is translated as **Cauchyness** of double categories:

$$\forall I \begin{array}{c} \xrightarrow{\lambda} \\ \perp \\ \xleftarrow{\rho} \end{array} J \quad \exists u \downarrow_J \quad (\lambda \dashv \rho) \cong (u_* \dashv u^*)$$

The function comprehensive completion is achieved as follows:

$$\begin{array}{ccc} \text{EEFib} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \text{EEFib}_{\text{funccomp}} \\ \text{Bil} \downarrow & \begin{array}{c} \text{Cau} \\ \perp \\ \xleftarrow{\quad} \end{array} & \uparrow \\ \text{CartEq} & & \text{CartEq}_{\text{Cauchy}} \end{array}$$

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Conclusion

- introduced cartesian fibrational virtual double categories as a relation-oriented framework for predicate logic.
- presented the $\mathbb{B}il$ -construction which produces a cartesian fibrational VDC from a cartesian fibration.

$$\begin{array}{ccc} & \mathbb{B}il & \\ & \rightarrow & \\ \text{EEFib} & & \text{CartEq}' \\ \downarrow & \lrcorner & \downarrow \\ \text{CartFib} & \rightarrow & \text{CartFVDC} \\ & \mathbb{B}il & \end{array}$$

Others in my thesis

20/20

- Recovering a characterization theorem in (Hoshino, Nasu, '23).
- Connection to bicategories.
- Self-duality on Frobenius equipments.
(• An internal logic of fibrational VDCs.)

Future Work

- develop theories of exact completion and other kinds of completion in terms of virtual double categories.
- ?

Reference

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Thank you!

my homepage :

hayatonasu.github.io

my thesis :

[arXiv:2501.17869](https://arxiv.org/abs/2501.17869)

