Categorical Logic meets Double Categories

Logic Winter School III

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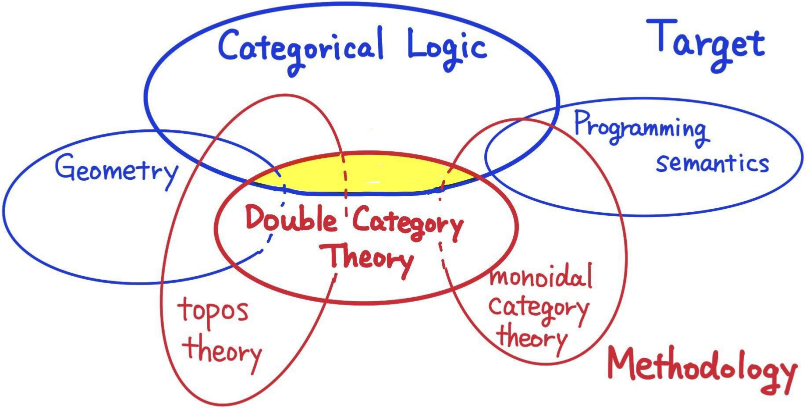
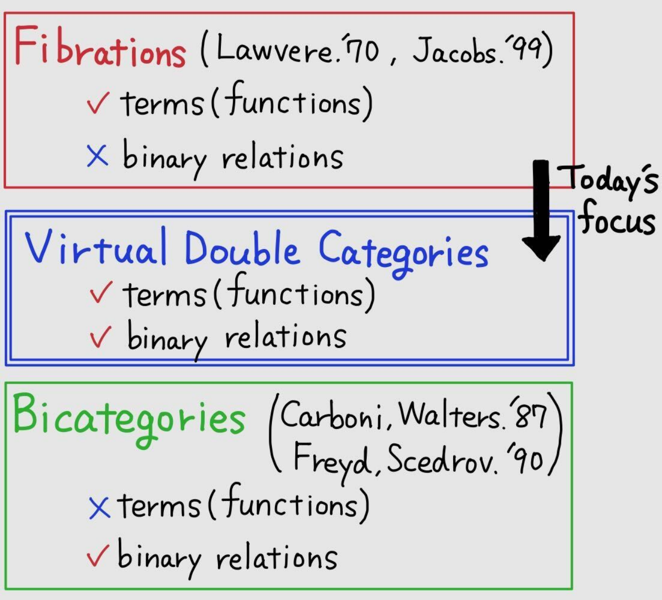
Outline



Categorical Logic
Target
Programming semantics
Geometry
Double Category Theory
topos theory
monoidal category theory
Methodology

Outline


Fibrations (Lawvere.'70, Jacobs.'99)
✓ terms (functions)
✗ binary relations
Today's focus
Virtual Double Categories
✓ terms (functions)
✓ binary relations
Bicategories (Carboni, Walters.'87 Freyd, Scedrov.'90)
✗ terms (functions)
✓ binary relations

Categorical Logic

Fibrations and Bicategories

Double Categories

Virtual Double Categories for predicate logic

Categorical Logic

Categorical Logic

Logical formulas are often interpreted with sets and functions, but categorical semantics generalizes these to objects and morphisms in general categories.

Example

$$(x + y) + z = x + (y + z)$$

$$\begin{tikzcd} A \times A \times A \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] \arrow[dr, phantom, "\circlearrowleft"] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"'] & A \end{tikzcd}$$

where

- A is a set /topological space/manifold
- m is a function /continuous map/smooth map

Equational logic can be interpreted in categories with finite products.

How about other logical symbols?

In set-theoretic semantics, $\exists x.\alpha(x)$ is interpreted as
" there exists an element $a \in A$ such that $\alpha(a)$."

$\iff$ the subset $\{\, a \mid \alpha(a) \,\} \subseteq A$ is not empty.

The interpretation of $\alpha(x_1, \ldots, x_n)$ is given as a subset
~~subset~~ subobject of the n-th power of A:
$[\![\alpha(x_1, \ldots, x_n)]\!] \subseteq A \times \cdots \times A$.

The interpretation of $\exists x.\alpha(x, y)$ is the image of $\pi_1 \circ i$:

$$
\begin{tikzcd}
{[\![\alpha(x,y)]\!]} \arrow[r, hook, "i"] \arrow[dr, two heads] & A \times A \arrow[r, "\pi_1"] & A \\
& {[\![\exists x.\alpha(x,y)]\!]} \arrow[ur, hook] &
\end{tikzcd}
\quad \text{in} \quad \mathcal{S}\mathrm{et}
$$

in a category with image factorization (a regular category)

Hierarchy of Logical Systems and Categories

Corresponding to the hierarchy of logical systems, we have a hierarchy of classes of categories (with structures).

$$
\begin{array}{c||c}
=, \wedge & \text{Categories with finite limits} \\
\shortmid & \cup\!| \\
=, \wedge, \exists \ \text{(regular logic)} & \text{Regular categories} \\
\shortmid & \cup\!| \\
=, \wedge, \vee, \exists, \forall, \neg \ \text{(first-order logic)} & \text{Heyting categories}
\end{array}
$$

Categorical logic is the study of logical phenomena within these and other categories.

Hierearchy of Logical Systems and Categorical Structures

One important direction of research is to enhance categories with structure for richer logical systems. This is called logical completion of categories (or categorical structures).

Example (Exact Completion)

regular logic	Regular categories
	$\uparrow \quad \downarrow$ Exact completion
regular logic with quotient types	Barr-exact categories

Different categorical structues

Different categorical structures have been employed in the development of categorical logic

- to include various kinds of semantics,
- to gain more flexibility in terms of the correspondence of logical systems and structures,
- to make some construction and proofs simple.

We will see two of them: fibrations and bicategories.

Fibrations and Bicategories

Using fibrations, one can generalize the interpretation of formulas from subobjects in a category to objects in the fibers of a fibration.

Example

Defining a fibration depending on your purpose, you can interpret formulas as

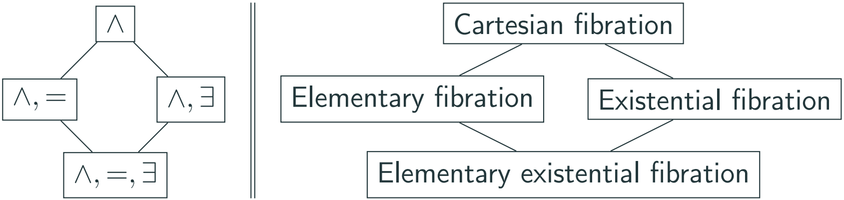
- subsets in $\mathcal{S}\mathrm{et}$, or subobjects in any category with finite limits,
- closed subsets in $\mathcal{T}\mathrm{op}$,
- pointwise values in H for a Heyting algebra H.

	in $\mathcal{B}$	in the fiber $\mathcal{E}_A$
Objects	where variables range over	formulas with a variable for A $\alpha(x) \quad (x \in A)$
Morphisms	sequences of terms	proofs of the implication $\alpha(x) \Rightarrow \beta(x)$

Fibrations and Logical Systems

The origin of the semantics on fibrations dates back to (Lawvere, 1970) and is widely accepted as a general categorical model of predicate logic (Jacobs, 1999).

We have the following hierarchy of classes of fibrations:



Recent studies on logical completions of categories are mostly based on fibrations ("$\supseteq$" doctrines).

- Quotient completion (Maietti & Rosolini, 2013a, 2013b)
- Existential completion (Trotta, 2020)

Bicategorical Approach

With bicategories, one can conceptually handle (binary) relations instead of predicates (formulas). The prototypical example is $\mathbf{Rel}$, consisting of sets, relations, and inclusions.

$$
\begin{tikzcd}
A \arrow[r, bend left, "R", "\shortmid"{marking}] \arrow[r, bend right, "S"', "\shortmid"{marking}] \arrow[r, phantom, "\subseteq"] & B
\end{tikzcd}
\quad\Bigg\|\quad
\begin{tikzcd}
R \arrow[d, hook] \arrow[dr, hook] & \\
& A \times B \\
S \arrow[ur, hook] &
\end{tikzcd}
$$

The identity relation :

$$A \xrightarrow{\delta_A} A \;=\{\,(a, a') \mid a = a'\,\}$$

The composition of relations :

$$A \xrightarrow{R} B \xrightarrow{R'} C \;= \left\{ (a, c) \;\middle|\; \exists b \in B.\ \begin{array}{l} (a, b) \in R \\ \wedge\ (b, c) \in R' \end{array} \right\}$$

Bicategorical Approach

There have been two main frameworks that axiomatize bicategories of this kind:

- Allegories (Freyd & Scedrov, 1990)
- Cartesian bicategories (Carboni, Kelly, Walters, & Wood, 2007; Carboni & Walters, 1987)

The bicategories $\mathbf{Rel}(\mathcal{B})$ of internal relations in regular categories are characterized in terms of either of these structures.

Pros and Cons of Bicateogrical Approach

Pros: Compositionality of relations is convenient for many constructions such as exact completion.

Cons: Lack of the notion of functions makes it difficult to interpret logical systems.

Double Categories

Definition

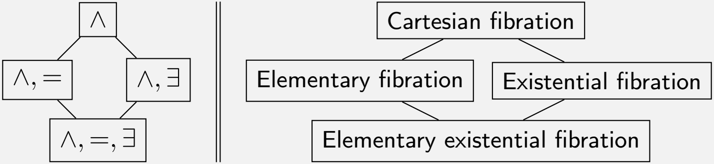
A double category $\mathbb{D}$ consists of the following data:

- objects $A, B, C, \ldots$
- tight arrows $\begin{array}{c} A \\ \downarrow f \\ B \end{array}, \begin{array}{c} B \\ \downarrow g \\ C \end{array}, \ldots$
- loose arrows $A \overset{R}{\nrightarrow} B$, $B \overset{S}{\nrightarrow} C$, $\ldots$
- cells
$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mu"] & B \arrow[d, "g"] \\ C \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}, \ldots$$

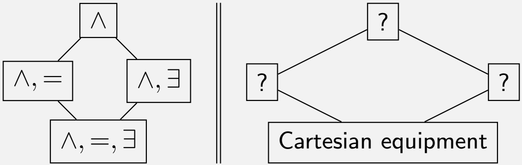
with compositions of tight arrows, loose arrows, and cells satisfying some axioms.

To interpret substitution and the conjunction, we need to consider cartesian equipments ($\simeq$ cartesian fibrational double categories) (Aleiferi, 2018).

Where does cartesian equipments fit in?



Where does cartesian equipments fit in?



Why? : The composition and the identities of loose arrows implicitly subsume the interpretability of $\exists$ and $=$.

Virtual Double Categories for predicate logic

Consider a cell

$$\begin{tikzcd} & B \arrow[dr, bend left, "S", "|" marking] & \\ A \arrow[ur, bend left, "R", "|" marking] \arrow[rr, bend right, "T"', "|" marking] \arrow[rr, phantom, "\rotatebox{90}{$\subseteq$}"] & & C \end{tikzcd}$$

in $\mathbf{Rel}$.

This corresponds to the following Horn clause.

$$\exists b \in B.\ (R(a,b)\ \wedge\ S(b,c))\ \Rightarrow\ T(a,c). \qquad (a \in A, c \in C)$$

Even without $\exists$ and $=$, we can still express this differently as follows.

$$R(a,b)\ \wedge\ S(b,c)\ \Rightarrow\ T(a,c). \qquad (a \in A, b \in B, c \in C)$$

A virtual double category is a structure based on generalized cells with n-ary inputs, not on the composition of loose arrows.

Virtual Double Categories

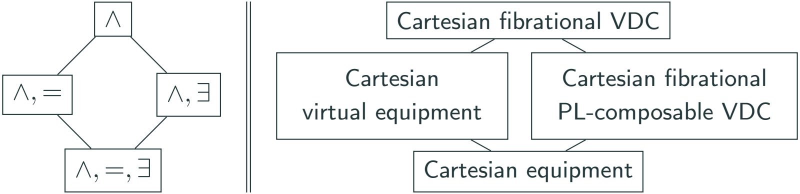
Definition

A virtual double category is … (See the appendix.)

Example

Any double category "is" a virtual double category.

Our contribution is to present the following correspondence.



If this talk caught your interest,

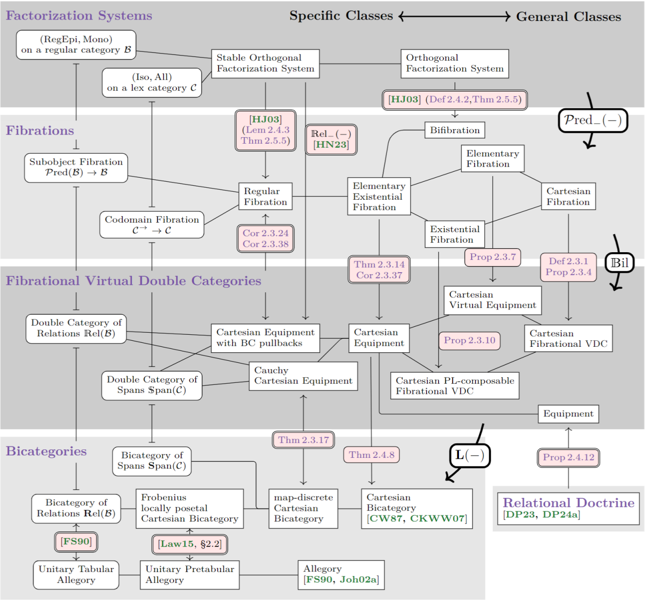
please take a look at:

- A more in-depth take :https://hayatonasu.github.io/hayatonasu/Talks/KCTM2025.pdf,
- My master's thesis: "Logical Aspects of Virtual Double Categories" https://arxiv.org/abs/2501.17869,
- Introduction to Categorical Logic: Lecture materials in Logic Winter School 2023 by Hisashi Aratake https://sites.google.com/view/logic-winter-school-2023

I will give a talk on the thesis but from a different perspective at CSCAT in Kumamoto:
https://hisashi-aratake.gitlab.io/event/cscat2025.html,

or wherever you would invite me to speak!

Thank you!



Factorization Systems
Specific Classes ⟷ General Classes
(RegEpi, Mono) on a regular category $\mathcal{B}$
Stable Orthogonal Factorization System
Orthogonal Factorization System
(Iso, All) on a lex category $\mathcal{C}$
[HJ03] (Def 2.4.2, Thm 2.5.5)
Fibrations
[HJ03] (Lem 2.4.3 Thm 2.5.5)
$\mathbb{R}\mathrm{el}_{-}(-)$ [HN23]
Bifibration
$\mathcal{P}\mathrm{red}_{-}(-)$
Elementary Fibration
Subobject Fibration $\mathcal{P}\mathrm{red}(\mathcal{B}) \to \mathcal{B}$
Regular Fibration
Elementary Existential Fibration
Cartesian Fibration
Codomain Fibration $\mathcal{C}^{\to} \to \mathcal{C}$
Cor 2.3.24 Cor 2.3.38
Existential Fibration
Prop 2.3.7
Thm 2.3.14 Cor 2.3.37
Def 2.3.1 Prop 2.3.4
$\mathbb{B}\mathrm{il}$
Fibrational Virtual Double Categories
Cartesian Virtual Equipment
Double Category of Relations $\mathbb{R}\mathrm{el}(\mathcal{B})$
Cartesian Equipment with BC pullbacks
Cartesian Equipment
Prop 2.3.10
Cartesian Fibrational VDC
Cauchy Cartesian Equipment
Double Category of Spans $\mathbb{S}\mathrm{pan}(\mathcal{C})$
Cartesian PL-composable Fibrational VDC
Equipment
Bicategories
Thm 2.3.17
Thm 2.4.8
$\mathbf{L}(-)$
Prop 2.4.12
Bicategory of Spans $\mathbf{Span}(\mathcal{C})$
Bicategory of Relations $\mathbf{Rel}(\mathcal{B})$
Frobenius locally posetal Cartesian Bicategory
map-discrete Cartesian Bicategory
Cartesian Bicategory [CW87, CKWW07]
Relational Doctrine [DP23, DP24a]
[FS90]
[Law15, §2.2]
Unitary Tabular Allegory
Unitary Pretabular Allegory
Allegory [FS90, Joh02a]

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What are the advantages of the framework of (virtual) double categories

- can deal with functions and relations as different entities, which benefits when we do not assume the unique choice principle.
- can be formulated in a simple way, without explicitly mentioning the Beck-Chevalley condition and the Frobenius laws (, which are somewhat artificial to define).

Virtual Double Categories

Definition

A virtual double category $\mathbb{D}$ consists of the following data:

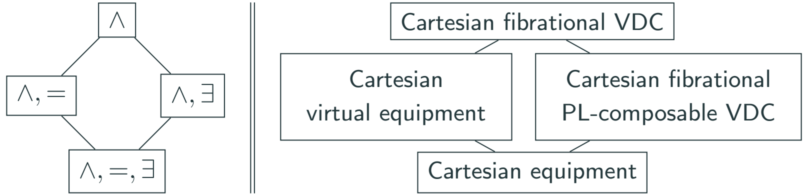
- objects $A, B, C, \ldots$
- tight arrows $\begin{array}{c} A \\ \downarrow f \\ B \end{array}, \begin{array}{c} B \\ \downarrow g \\ C \end{array}, \ldots$
- loose arrows $A \xrightarrow{R} B$, $B \xrightarrow{S} C$, ...
- virtual cells

$$\begin{tikzcd} A \arrow[r, "R"] \arrow[d, "f"'] & B \arrow[r, "R'"] \arrow[d, phantom, "\mu"] & C \arrow[d, "g"] \\ D \arrow[rr, "S"'] & & E \end{tikzcd}, \ldots$$

with compositions of tight arrows, ~~loose arrows~~ and virtual cells satisfying some axioms.

Virtual Double Categories (VDCs)

Our contribution is to present the following correspondence.



More precisely, we compare VDCs with fibrations, which is firmly established as a categorical basis for predicate logic.

In Def 2.3.1, we construct a cartesian fibrational VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ from a cartesian fibration $\mathfrak{p}$, and present this as a 2-functor $\mathbb{B}\mathrm{il} \colon \mathbf{Fib_{cart}} \to \mathbf{FVDbl_{cart}}$. This is a generalization of $\mathbb{F}\mathrm{r}$-construction in (Shulman, 2008) and $\mathbb{M}\mathrm{atr}$ in (Lawler, 2015).

Main Results

Theorem (Theorem 2.3.17)

The 2-category of elementary existential fibrations is a pullback of $\mathbb{B}\mathrm{il}$ *along the forgetful 2-functor from* $\mathbf{Eqp_{cart}}$.

$$\begin{tikzcd} \mathbf{EEF} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Eqp_{cart}} \arrow[d] \\ \mathbf{Fib_{cart}} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \mathbf{FVDbl_{cart}} \end{tikzcd}$$

This indicates that "a cartesian fibration $\mathfrak{p}$ can interpret regular logic if and only if the cartesian fibrational virtual double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is actually a cartesian fibrational double category ($\simeq$ cartesian equipment)."

A simple consequence

Example

For a category with finite limits $\mathcal{B}$, we have a cartesian fibration $\mathcal{S}\mathrm{ub}(\mathcal{B}) \to \mathcal{B}$.

It is an elementary existential fibration if and only if $\mathcal{B}$ is regular. (Jacobs, 1999)

Using the theorem, $\mathbb{B}\mathrm{il}(\mathcal{S}\mathrm{ub}(\mathcal{B})) = \mathbb{R}\mathrm{el}(\mathcal{B})$ is a cartesian equipment if and only if $\mathcal{B}$ is regular.

Other Results

We further obtain the following results.

- We show that $\mathbb{B}\mathrm{il}\colon \mathbf{EEF} \to \mathbf{Eqp_{cart}}$ is locally an equivalence and characterize the essential image of $\mathbb{B}\mathrm{il}$ by so-called the Frobenius axiom (Hoshino & Nasu, 2023; Lambert, 2022; Walters & Wood, 2008) (Cor. 2.3.37).
- This restricts to a biequivalence $\mathbb{B}\mathrm{il}\colon \mathbf{RegFib} \to \mathbf{Eqp_{BC}}$ (Cor. 2.3.38).
- We revisit the main result of (Hoshino & Nasu, 2023), which characterizes the double categories $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{B})$ for an SOFS (E,M), in terms of the $\mathbb{B}\mathrm{il}$-construction and the characterization of the fibrations of M-subobjects $\mathcal{M} \to \mathcal{B}$ (Hughes & Jacobs, 2003) (Cor. 2.5.8).

Other Results

The connection to the bicategorical approach is also discussed.

- We show that the loose bicategory of a cartesian equipment is a cartesian bicategory in the sense of (Carboni et al., 2007) (Thm. 2.4.8).
- We show that the loose bicategory of a Frobenius cartesian equipment is a self-dual compact closed bicategory in the sense of (Stay, 2016) (Prop. 2.3.30).
- We recover the adjunction between the category of elementary existential doctrines and the category of (Frobenius) locally-posetal cartesian bicategories (Bonchi, Santamaria, Seeber, & Sobociński, 2021) via double categories (Rem. 2.5.21).

$$
\begin{tikzcd}
\mathbf{Fib}_{\times\wedge=\exists} \arrow[r, shift left, "\mathbb{Bil}"] \arrow[r, phantom, "\simeq"] & \mathbf{Eqp}_{\mathbf{Frob}} \arrow[l, shift left, "\mathfrak{uni}"] \arrow[r, shift left, "\mathbb{Cau}"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left, "\mathbf{L}(-)"] & \mathbf{Eqp}_{\mathbf{Frob},\mathbf{Cauchy}} \arrow[l, hook, shift left] \arrow[r, shift left, "\mathbf{L}(-)"] \arrow[r, phantom, "\simeq"] & \mathbf{CartBi}_{\mathbf{Frob},\mathbf{MD}} \arrow[l, shift left, "\mathbb{Map}(-)"]
\end{tikzcd}
$$