Structural Set Theory
— Towards SEFAR —

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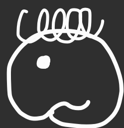
導入

- 数学の基礎
  - 集合論
    - Material Set Theory
      - ZFC
      - BZF
    - Structural Set Theory
      - ETCS
      - SEAR
  - 型理論
  - ⋮

"ZFC has one major flaw: its use of the word 'set' conflicts with how most mathematicians use it."

(Leinster, "Rethinking set theory", 2012)

$3 \in 5$, $7 \notin \pi$,
$n+1 = n \cup \{n\}$



…

Does it matter?

A structural set theorist

構成

1. Structural set theory とは？

2. SE(F)AR

3. SEFAR as an internal logic

構成

1. Structural set theory とは？

2. SE(F)AR

3. SEFAR as an internal logic

注意

- 本講演は，nLab 及び n-Category Café への Micheal Shulman の posts がもとになっています．
- structural set theory は嫌いでも圏論のことは嫌いにならないで下さい．

1. Structural set theory とは？
2. SE(F)AR
3. SEFAR as an internal logic

Structural set theory の経緯

Lawvere "An elementary theory of the category of sets" (1964)

- 集合の圏の特徴付け ⤳ ETCS
- 圏論による数学の基礎付け

現代的には, well-pointed topos with NNO and AC を指す.

その後, Mike Shulman が structuralism を元に名付けた.

他には, Local set theory, SEAR, structural ZFC,

HoTT における集合も Structural set theory の一つ.

Structural とは?

"集合を「何を含むか」ではなく、
「他の集合とどう相互作用するか」で規定する."

例　ZFCでは、$\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots\}$ と定義される.

ここで ω の elements として何が含まれるかではなく、
ω が帰納的な性質をもつことが ω の役割として重要.

$\rightsquigarrow$ natural number object N in a category SET

$$
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "a"'] & N \arrow[r, "s"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[d, "\varphi\ \exists!"] \\
 & A \arrow[r, "f"'] & A
\end{tikzcd}
$$

in SET

ETCS の公理 (Formally)

圏の (first-order) theory を用いて完全に形式化が与えられる.

fully-formal (non-categorical) ETCS	ZF
2つのソート (o, m) (集合と関数)	1つのソート (集合)
dom : m → o, cod : m → o, id : o → m, ∘ : m, m → m	∈ : binary relation
power object $\ulcorner \forall \begin{array}{ccc} S & \overset{\forall}{\rightarrowtail} & Y \times X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \exists!\, f \times \mathrm{id}} \\ \in_X & \underset{\text{given}}{\rightarrowtail} & PX \times X \end{array} \urcorner$	power set axiom $\forall X\ \exists Y\ \forall Z$ $Z \in Y$ $\leftrightarrow \forall W(W \in Z \to W \in X)$

ETCSの問題点とその解決

- 1970年代の Cole, Mitchell, Osius らの結果として
  ETCS (w/o AC) と BZF (ZF but only Δ_0-replacement) の
  等価性が知られている. (cf. [SGL, VI §10])

$$\begin{tikzcd}
\text{ETCSのモデル} \arrow[r, bend left, "\text{pure-sets}"] & \text{BZFのモデル} \arrow[l, bend left, "\text{Set 圏}"]
\end{tikzcd}$$

- これは structural set theory の限界を示すものではない.
  対象全体にわたる量化がトポス internal にできなかったが…

ETCSの問題点とその解決

- ETCS + Structural collection $\leftrightarrow$ ZF (S2017)
  対象の量化を許した collection を付加した体系.
  (internal logic ではない.)
- Stack semantics (S 2010 and ...)
  object (set) にわたる量化の解釈の方法を与えた.
  (これは internal logic (Kripke-Joyal semantics) の拡張.)

equiconsistency の強弱は structural set theory そのものの問題ではなく, topos internal logic の問題.

1. Structural set theory とは？

2. SE(F)AR

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SEAR

SEAR (Sets, Elements, and Relations) は ETCS と一線を画する structural set theory の一つ. (nLab page by Shulman et.al.)

特徴

- elements を基本的な概念として含む.
- category に関与しない.
- separation がほぼ公理として表われる.
- ZF と等価.

material set theory

ZF

NBG

structural set theory

SEAR

structural ZFC

categorical set theory

ETCS

SEAR

SEAR は 3 つの dependent sorts からなる.

ソート：

$$\frac{}{\mathrm{Set} : \mathrm{Sort}} \qquad \frac{A : \mathrm{Set}}{\mathrm{Ele}_A : \mathrm{Sort}} \qquad \frac{A, B : \mathrm{Set}}{\mathrm{Rel}_{A,B} : \mathrm{Sort}}$$

原子論理式：

$$\frac{x, y : \mathrm{Ele}_A}{x = y : \mathrm{Formula}} \qquad \frac{\varphi, \psi : \mathrm{Rel}_{A,B}}{\varphi = \psi : \mathrm{Formula}}$$

$$\frac{x : \mathrm{Ele}_A , \; y : \mathrm{Ele}_B , \; \varphi : \mathrm{Rel}_{A,B}}{\varphi(x, y) : \mathrm{Formula}}$$

ここから logical connectives / quantifiers (including sets) によってえられる formula を考える. (必要なら context を考慮する.)

SEAR

Axiom 0 $\exists A : \mathrm{Set}\ \exists a : \mathrm{Ele}_A\ \top$

Axiom 1 (Relational comprehension)

For each $\varphi(x,y)$: formula $[A,B : \mathrm{Set},\ x : \mathrm{Ele}_A,\ y : \mathrm{Ele}_B]$,

$$\forall A,B : \mathrm{Set}\ \ \exists!\, p : \underbrace{A \looparrowright B}_{\substack{\equiv\\ \mathrm{Rel}_{A,B}}}\ \forall x : \mathrm{Ele}_A\ \forall y : \mathrm{Ele}_B\ \ p(x,y) \leftrightarrow \varphi(x,y)$$

Function $f : A \to B$ は functional relation $f : A \looparrowright B$ として

定義される. 以降, $\dfrac{x : \mathrm{Ele}_A \quad f : A \to B}{f(x) : \mathrm{Ele}_B}$ と仮想的な term を使う.

Axiom 2 (Tabulation)

$\forall A, B : \mathrm{Set} \;\; \forall p : A \nrightarrow B$

$\exists\, |p| : \mathrm{Set} \;\; \exists f : |p| \to A \;\; \exists g : |p| \to B \qquad (1) \wedge (2)$

where

$$(1) \;\; \forall x : \mathrm{Ele}_A \;\; \forall y : \mathrm{Ele}_B \left(p(x, y) \leftrightarrow \exists r : \mathrm{Ele}_{|p|} \begin{array}{l} f(r) = x \;\; \wedge \\ \quad g(r) = y \end{array} \right)$$

$$(2) \;\; \forall r, s : \mathrm{Ele}_{|p|} \;\; \big(f(r) = f(s) \wedge g(r) = g(s) \big) \to r = s$$

$$\begin{tikzcd} & {|p|} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "p"', "\shortmid" marking] & & B \end{tikzcd}$$

$$|p| =: \{ x : \mathrm{Ele}_A ,\, y : \mathrm{Ele}_B \mid p(x, y) \}$$

Axiom 3 : 巾集合 , Axiom 4 : 無限 (割愛)

Axiom 5 : collection

$\forall A : \mathrm{Set}\ \exists B : \mathrm{Set}\ \exists f : B \to A\ \exists Y : \mathrm{Set}\ \exists M : B \nrightarrow Y$ (1) $\wedge$ (2)

for each $\varphi(a, X)$: formula $[X, A : \mathrm{Set}, a : A]$

where

(1) $\forall b : \mathrm{Ele}_B\ \ \varphi(f(b), \underline{M_b})$ $\{ y : \mathrm{Ele}_Y \mid M(b, y) \}$

(2) $\forall a : \mathrm{Ele}_A\ \forall X : \mathrm{Set}\ \big(\varphi(a, X) \to \exists b : \mathrm{Ele}_B\ f(b) = a \big)$

SEAR

Theorem [nLab]

SEAR is equiconsistent with ZF.

ZF の model ⇝ SEAR の model : Set, Ele_A, $\mathrm{Rel}_{A,B}$ として自然に定義される.

SEAR の model ⇝ ZF の model :

pure sets = 整礎外延的到達可能グラフの同型類

として構成される. (大変だが ETCS よりかは簡潔)

SEARの欠点

- functionをprimitiveな概念として持っていない.
  これは数学で現れるoperationとしての機能を果たしていない.

- predicativeな設定への拡張が難しい.
  functional relationとfunctionを区別する術がない.

$\rightsquigarrow$ functionsをprimitiveに追加すれば?

SEFAR

(cf. [nLab] structural ZFC, SEAR + ε)

SEFAR (Sets, Elements, Functions, and Relations) は SEAR に function の sort を追加したもの.

その際, SEAR との等価性のために,

Unique choice $\forall \phi : A \leftrightarrow B \ (\ulcorner \phi \text{ is a functional relation} \urcorner \to$
$\exists f : A \to B \ \forall x : \mathrm{Ele}_A \ \forall y : \mathrm{Ele}_B \ \phi(x, y) \leftrightarrow y = f(x) \ )$

が必要.

Proposition

SEFAR is equiconsistent with SEAR.

1. Structural set theory とは?

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As an internal logic

SEARは category の言葉で記述されていないが、
普遍性, 随伴, etc.

それを集合と関係のなす locally-posetal 2-category の internal logic と捉えることもできる。

> **Proposition**
>
> SEARの model S に対し、S における sets, elements, relations, implications によって、tabular allegory with units を得る。

As an internal logic

Categorical な presentation を SE(F)AR に与えるとすれば…

axiomatic structural set theory	fully-formal ETCS	SEAR	SEFAR
category-theoretic set theory	ETCS (topos)	tabular allegory with … [FS90, Joh02]	double category of relations with … [Lam21, HN23]

$$\begin{tikzcd} & A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & N \arrow[l, "t"'] \\ & B & \end{tikzcd}$$

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B$$

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\cdot"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$$

As an internal logic

ETCS と SEAR (allegorical set theory) の等価性を裏付けるのが

Theorem [Joh02, A3.2.10, 4.7]

$\mathcal{A}$: tabular allegory with units

$\Rightarrow \mathcal{A} \simeq \mathcal{R}el(\mathrm{Map}(\mathcal{A}))$ $\quad$ ($\mathrm{Map}(\mathcal{A})$: regular category)

If this holds, $\mathrm{Map}(\mathcal{A})$ is a topos $\iff$ $\mathcal{A}$: power allegory.

これの double category としての analogy が

Theorem [Lam21, HN23]

$\mathbb{D}$: locally posetal, Cauchy double category

$\Rightarrow \mathbb{D} \simeq \mathbb{R}el(\mathbb{D}_0)$ $\quad$ ($\mathbb{D}_0$: regular category)

Future work

- Predicative version of SEFAR の構築.
- Minimalist Foundation との関係.
- Stack semantics for double categories.
- Realizability interpretation の double categorical approach

interpret double categorically?

Logic (arithmetic)

interpret by KJ semantics

$\boxed{\text{categories of assemblies}}$ (Ass, ?)

$\downarrow$ ex/reg completion

recover

$\boxed{\text{realizability toposes}}$ $(\mathrm{Ass}_{ex/reg}, j)$

Thank you!

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Reference list

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Structural set theory の定式化

何をもって structural とするかは議論の余地がある.

"Def" A set theory $\mathcal{S}$ is structually presentable iff

it is presented as a notion of sets, it satisfies

dependent-sorted $(\mathbb{S}\mathrm{et}, \mathbb{E}\mathrm{lement}_\bullet)$- first order theory

for each φ : formula $[\Gamma, A : \mathbb{S}\mathrm{et}, x : \mathbb{E}\mathrm{lement}_A]$

$$\boxed{\begin{aligned}&\forall \vec{z} : \Gamma\ \forall A : \mathbb{S}\mathrm{et}\ \forall x, y : \mathbb{E}\mathrm{lement}_A \\ &\quad \text{is-set}(A) \to x \in A \to y \in A \to (\varphi \leftrightarrow \varphi[y/x])\end{aligned}}$$

holds. (nLab authors, structurally presented set theory)

Structural set theory の定式化

注意 : structurally presented かどうかは
a notion of sets としての表示に依存する.

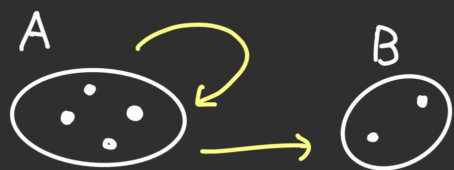
例 : • ZF is not structurally presented :

$$\varphi \equiv \emptyset \in x \quad [A : \mathbb{S}\mathrm{et},\ x : \mathbb{E}\mathrm{lement}_A]$$

$(A = \{\emptyset, \{\emptyset\}\},\ x = \emptyset,\ y = \{\emptyset\})$ などが反例.

• ETCS is structurally presented with appropriate presentation.

集合それ自体で質的に
2元の区別はできないということ.



SEFAR

SEFAR = SEAR +

Sorts : $A, B : \mathrm{Set} \mid \mathrm{Fun}_{A,B}$: Sort

Function symbols : $\mathrm{ev}_{A,B} : \mathrm{Ele}_A \times \mathrm{Fun}_{A,B} \to \mathrm{Ele}_B$

$\mathrm{id}_A : \to \mathrm{Fun}_{A,A}$

$\mathrm{comp}_{A,B,C} : \mathrm{Fun}_{B,C} \times \mathrm{Fun}_{A,B} \to \mathrm{Fun}_{A,C}$

（省略して $f : A \to B$, $f(x) : \mathrm{Ele}_B$, $f \circ g : A \to C$ などと書く.）

原子論理式 :

$$\frac{f, g : A \to B}{f = g \ : \mathrm{Formula}}$$

SEFAR

Additional axioms :

Function Extensionality $f(x) = g(x) \to f = g$

Evaluation of composites $(f \circ g)(x) = f(g(x))$

Evaluation of identities $id_A(x) = x$

Unique choice $\forall \phi : A \nrightarrow B \; (\ulcorner \phi \text{ is a functional relation} \urcorner \to$

$\exists f : A \to B \;\; \forall x : Ele_A \; \forall y : Ele_B \;\; \phi(x, y) \leftrightarrow y = f(x) \;)$

Proposition

SEFAR is equiconsistent with SEAR.