A Formal Theory of Anticolimits

Hayato Nasu

Kyoto University

hnasu@kurims.kyoto-u.ac.jp
hayatonasu.github.io

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Introduction

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C & {} \arrow[r, squiggly, "\exists ?", "\text{Taking colimit}"'] & {} & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & C \arrow[d] & {} & {} \arrow[l, squiggly, "\exists ?"', "\text{Taking anticolimit}"] & & C \arrow[d] \\
B & & & & B \arrow[r] & D & & & B \arrow[r] & D
\end{tikzcd}$$

The central question is:

"How can we know if a cocone is a colimit of some diagram?"

(Tataru, Vicary. "The theory and applications of anticolimits" (2024))

So many examples are out there!

For categories, 2-categories, additive categories, ...

Goal : A conceptual understanding of anticolimits.

Structure

1. Elements of anticolimit.
2. Formal category theory via (virtual) double categories.
3. Formal theory of anticolimits : general theory
4. Formal theory of anticolimits : application

The slides for today.

Structure

1. Elements of anticolimit.
2. Formal category theory via (virtual) double categories.
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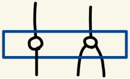
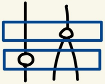
4. Formal theory of anticolimits : application

Original work

- Tataru and Vicary introduced anticolimits in the study of homotopy.io.

 is a "colimit" of 

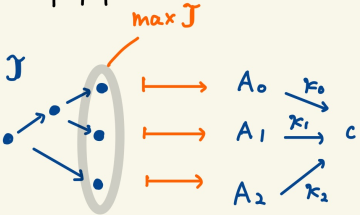
- An anticolimit of κ,

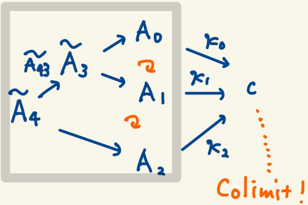
where
$$\begin{cases} J : \text{poset}, \\ c \in \mathcal{C} \\ A : \max J \longrightarrow \mathcal{C}, \\ \kappa : A \Rightarrow \Delta_c \end{cases}$$

is an extension $\tilde{A} : J \to \mathcal{C}$ of A

such that κ induces

a colimit cocone of $\tilde{A}$.





A recipe for anticolimits

J : poset, $X : \max \mathcal{J} \longrightarrow \mathcal{C}$, $c \in \mathcal{C}$, $\kappa : X \Rightarrow \Delta_c$

Def $\Pi_{\mathcal{J}}(\kappa) : \mathcal{J} \longrightarrow \mathcal{C}$ is defined (if possible) as follows :

- $j \in \mathcal{J}$ is mapped to a multiple pullback of $(X_i \xrightarrow{\kappa_i} c)_{i \geq j}$.
- $j \to j'$ in $\mathcal{J}$ is mapped to the canonical arrow in $\mathcal{C}$.

$\longmapsto$

$\mathcal{C}\rfloor$

$$\begin{tikzcd}
 & & X_0 \arrow[dr, "\kappa_0"] & \\
 & X_0 \times_c X_1 \arrow[ur] \arrow[r] & X_1 \arrow[r, "\kappa_1"'] & c \\
X_0 \times_c X_1 \times_c X_2 \arrow[ur] \arrow[rr] & & X_2 \arrow[ur, "\kappa_2"'] &
\end{tikzcd}$$

Theorem [TV24] If $\Pi_{\mathcal{J}}(\kappa)$ exists and κ has an anticolimit, then $\Pi_{\mathcal{J}}(\kappa)$ is an anticolimit of κ.

When $\mathcal{C}$ has enough limits, whether κ has an anticolimit can be checked just by looking at $\Pi_{\mathcal{J}}(\kappa)$.

Further Examples

① Regular epi

Def $f : A \to B$ in a category $\mathcal{C}$ is a regular epimorphism if it is a coequalizer of some arrows.

$$\begin{tikzcd} X \arrow[r, shift left, "k"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Prop If $\mathcal{C}$ has pullbacks, f is a regular epimorphism iff it is a coequalizer of its kernel pair.

$\in$ an effective epimorphism

$$\begin{tikzcd} A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

② Subcanonicity of sites

Def A site $(\mathcal{C}, T)$ is subcanonical if every representable presheaf is a sheaf w.r.t. T.

Prop TFAE when $\mathcal{C}$: complete.

(i) $(\mathcal{C}, T)$ is subcanonical.

(ii) For any T-covering $(A_i \xrightarrow{\kappa_i} C)_i$, C is a colimit of

$$\{\kappa_i\} \xrightarrow{\text{f.f.}} \mathcal{C}/C \xrightarrow{\mathrm{dom}} \mathcal{C}$$

(iii) For any T-covering $(A_i \xrightarrow{\kappa_i} C)_i$, C is a colimit of

$$\left(\begin{tikzcd} & A_i \times_C A_j \arrow[dl] \arrow[dr] & \\ A_i & & A_j \end{tikzcd}\right)_{i,j}.$$

Further Examples

③ Normal epi in Ab-cats

Def $f : A \to B$ in an Ab-cat. $\mathcal{C}$

is a normal epimorphism if it is a cokernel of some arrow.

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Prop In a finitely complete Ab-cat, an arrow is a normal epimorphism if it is a cokernel of its kernel.

④ Localization in 2-categories

Def An 1-cell is a localization if it is a coinverter of some 2-cell.

Prop A functor $F : \mathcal{C} \to \mathcal{D}$ is a localization iff

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, "\cong"] \\ {} \arrow[ur, phantom, "\circlearrowright"] & \mathcal{D} \end{tikzcd}$$

with $W := \{ f \mid Ff : \text{iso} \}$.

⑤ Effective tabulator in double cats

(Strong)

Prop In a flat double cat $\mathbb{D}$ with tabulators,

\# Cells ≤ 1 for each frame

$A \overset{p}{\nrightarrow} B$ is presented as

$$\begin{tikzcd} & \overset{\exists}{C} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "p"', "\shortmid" marking] & \text{opc.} & B \end{tikzcd}$$

if it has an effective tabulator

$$\begin{tikzcd} & \{p\} \arrow[dl, "l_p"'] \arrow[dr, "r_p"] & \\ A \arrow[rr, "p"', "\shortmid" marking] & \text{opc.} & B \end{tikzcd}$$

.

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Interlude: Formal Category Theory

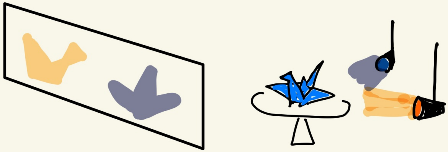
Formal Category Theory = Category Theory of Category Theories

Conceptual treatment from an abstract viewpoint

- $\mathcal{V}$-enriched category theory
- S-internal category theory
- S-fibred category theory

It studies how one can develop category theory inside ~~2-categories~~ by imagining it as the ~~2-category~~ of categories. (Virtual) double cats

Mathematical Phenomena	Categorical Treatment of …	Categories
Categorical Phenomena	Formal theory of …	VDCs (or other structures)

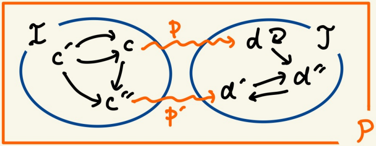


Goal : A formal theory of anticolimits.

Profunctors and Virtual equipments

Def A profunctor $P : \mathcal{I} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{J}$ is a functor $P : \mathcal{I}^{op} \times \mathcal{J} \to \mathrm{Set}$.

⚠ Contravariant on its domain.



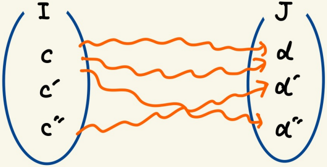
Prop The following correspond bijectively.

(i) Profunctos $P : \mathcal{I} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{J}$

(ii) Pairs of embeddings $\mathcal{I} \overset{i}{\hookrightarrow} \mathcal{P} \overset{j}{\hookleftarrow} \mathcal{J}$

s.t. $\mathrm{ob}\,\mathcal{I} \sqcup \mathrm{ob}\,\mathcal{J} \xrightarrow[\cong]{\langle i,j \rangle} \mathrm{ob}\,\mathcal{P}$ and $\mathcal{P}(j(d), i(c)) = \emptyset \quad (\forall d, c)$

Ex
- For a category $\mathcal{C}$, we have the hom-profunctor $\mathcal{C}(-,\bullet) : \mathcal{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{C}$.
- For two sets I, J seen as discrete categories, a profunctor $I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ is a bipartite graph (or span).



Profunctors and Virtual equipments (continued)

A natural trans.
$$\begin{tikzcd} & \mathcal{C} \arrow[dd, bend right=60, "F"'] \arrow[dd, bend left=60, "G"] \arrow[dd, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\ & & \\ & \mathcal{D} & \end{tikzcd}$$
is a natural family $(\alpha_c \in \mathcal{D}(Fc, Gc))_{c\in\mathcal{C}}$

↓ generalize

Naturality only involves the structure of the hom-profunctor $\mathcal{D}(-,\bullet)$.

A natural trans.
$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\alpha"] & \\ \mathcal{D} \arrow[rr, "P"', "\shortmid"{marking}] & {} & \mathcal{D}' \end{tikzcd}$$
is a natural family $\left(\underline{\underline{\alpha_c \in P(Fc, Gc)}}\right)_{c\in\mathcal{C}}$

$(\alpha_{c,c'} : \mathcal{C}(c,c') \to P(Fc, Gc'))_{c,c'}$

$$\begin{tikzcd} & \mathcal{C} \arrow[r, "\mathcal{C}(-,\bullet)", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathcal{C} \arrow[d, "G"] \\ \leftrightsquigarrow & \mathcal{D} \arrow[r, "P"', "\shortmid"{marking}] & \mathcal{D} \end{tikzcd}$$

↓ generalize

A natural trans.
$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "Q_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "Q_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \mathcal{C}_n \arrow[d, "G"] \\ \mathcal{D} \arrow[rrr, "P"', "\shortmid"{marking}] & & & \mathcal{D}' \end{tikzcd}$$
is a natural family

$$\left(Q_0(c_0, c_1) \times \cdots \times Q_n(c_{n-1}, c_n) \xrightarrow{\alpha} P(Fc_0, Gc_n)\right)_{c_0,\dots,c_n}$$

Profunctors and Virtual equipments (continued)

A natural trans.

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "Q_1", "|"{marking}] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "|"{marking}] & \cdots \arrow[r, "Q_n", "|"{marking}] & \mathcal{C}_n \arrow[d, "G"] \\
\mathcal{D} \arrow[rrr, "P"', "|"{marking}] \arrow[rrru, phantom, "\alpha"] & & & \mathcal{D}'
\end{tikzcd}$$

is a natural family

$$\left(Q_0(c_0, c_1) \times \cdots \times Q_n(c_{n-1}, c_n) \xrightarrow{\alpha} P(Fc_0, Gc_n)\right)_{c_0,\dots,c_n}$$

These natural transformations can be composed like

$$\begin{tikzcd}
{} \arrow[r, "|"{marking}] \arrow[d] \arrow[drr, phantom, "\alpha_1"] & \cdots \arrow[r, "|"{marking}] & {} \arrow[d] & \cdots & {} \arrow[r, "|"{marking}] \arrow[d] \arrow[drr, phantom, "\alpha_n"] & \cdots \arrow[r, "|"{marking}] & {} \arrow[d] \\
{} \arrow[rr, "|"{marking}] \arrow[d] \arrow[drrrrrr, phantom, "\beta"] & & {} & \cdots & {} \arrow[rr, "|"{marking}] & & {} \arrow[d] \\
{} \arrow[rrrrrr, "|"{marking}] & & & & & & {}
\end{tikzcd}$$

,

and constitute a virtual double category $\mathbb{P}$ROF.

Def A virtual double category $\mathbb{X}$ consists of

- a category $\mathbb{X}_t$ of objects and tight arrows.

$$\begin{tikzcd}
\bullet \arrow[d, "f"] \\
\bullet
\end{tikzcd}$$

- a family of classes of loose arrows $(\mathbb{X}(x, x'))_{x, x' \in \mathbb{X}}$

$$\begin{tikzcd}
\bullet \arrow[r, "p", "|"{marking}] & \bullet
\end{tikzcd}$$

- a family of classes of cells for each frame

$$\begin{tikzcd}
\bullet \arrow[r, "p_1", "|"{marking}] \arrow[d, "f"'] & \cdots \arrow[r, "p_n", "|"{marking}] & \bullet \arrow[d, "g"] \\
\bullet \arrow[rr, "q"', "|"{marking}] \arrow[rru, phantom, "\alpha"] & & \bullet
\end{tikzcd}$$

.

- Data of composition and identities.

Profunctors and Virtual equipments (continued)

Def A restriction of

$$\begin{tikzcd} I \arrow[d, "f"'] & J \arrow[d, "g"] \\ I' \arrow[r, "|" marking, "p"'] & J' \end{tikzcd} \quad \text{is a cell} \quad \begin{tikzcd} I \arrow[r, "|" marking, "{p[f;g]}"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{rest}"] & J \arrow[d, "g"] \\ I' \arrow[r, "|" marking, "p"'] & J' \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} K \arrow[r, "|" marking] \arrow[d, "k"'] & \cdots \arrow[r, "|" marking] & L \arrow[d, "h"] \\ I \arrow[d, "f"'] & \forall \quad \alpha & J \arrow[d, "g"] \\ I' \arrow[rr, "|" marking, "p"'] & & J' \end{tikzcd} = \begin{tikzcd} K \arrow[r, "|" marking] \arrow[d, "k"'] & \cdots \arrow[r, "|" marking] & L \arrow[d, "h"] \\ I \arrow[rr, "|" marking, "{p[f;g]}"'] \arrow[d, "f"'] & \exists! \, \tilde{\alpha} & J \arrow[d, "g"] \\ I' \arrow[rr, "|" marking, "p"'] & \text{rest} & J' \end{tikzcd}.$$

Def A (loose) unit on I is

a loose arrow $U_I : I \nrightarrow I$

together with a cell

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, equal] & \\ I \arrow[rr, "|" marking] & \eta_I & I \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} K_n \arrow[r, "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "|" marking] & K_1 \arrow[r, "|" marking] & I \arrow[r, "|" marking] & L_1 \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] & L_m \arrow[d, "g"] \\ N \arrow[rrrrrr, "|" marking, "p"'] & & \forall & & \alpha & & M \end{tikzcd}$$

$$\|$$

$$\begin{tikzcd} K_n \arrow[r, "|" marking] \arrow[d, equal] & \cdots \arrow[r, "|" marking] & K_1 \arrow[r, "|" marking] \arrow[d, equal] & & I \arrow[rr, "|" marking] \arrow[dl, equal] \arrow[dr, equal] & & L_1 \arrow[r, "|" marking] \arrow[d, equal] & \cdots \arrow[r, "|" marking] & L_m \arrow[d, equal] \\ K_n \arrow[r, "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "|" marking] & K_1 \arrow[r, "|" marking] & I \arrow[rr, "|" marking] & \eta_I & I \arrow[r, "|" marking] & L_1 \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] & L_m \arrow[d, "g"] \\ N \arrow[rrrrrrrr, "|" marking, "p"'] & & & & \exists! \, \tilde{\alpha} & & & & M \end{tikzcd}$$

with the cells $\mathrm{id}, \cdots, \mathrm{id}, \eta_I, \mathrm{id}, \cdots, \mathrm{id}$ in the middle row.

Def A virtual equipment is

a virtual double categories with restrictions and units on every object.

Colimits via profunctors

Natural transformations of the form

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "P", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathcal{J} \arrow[d, "H"] \\
\mathcal{C} \arrow[r, "\mathcal{C}(-,\bullet)"', "\shortmid" marking] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{I} \arrow[rr, "P", "\shortmid" marking] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\alpha"] & \mathcal{J} \arrow[dl, "H"] \\
& \mathcal{C} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "P", "\shortmid" marking] \arrow[dr, "F"'] & \mathcal{J} \arrow[r, "Q", "\shortmid" marking] \arrow[d, phantom, "\beta"] & \mathcal{K} \arrow[dl, "G"] \\
& \mathcal{C} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{I} \arrow[r, "P", "\shortmid" marking] \arrow[d, "F"'] & \mathcal{J} \arrow[r, "Q", "\shortmid" marking] \arrow[d, phantom, "\beta"] & \mathcal{K} \arrow[d, "G"] \\
\mathcal{C} \arrow[rr, "\mathcal{C}(-,\bullet)"', "\shortmid" marking] & & \mathcal{C}
\end{tikzcd}$$

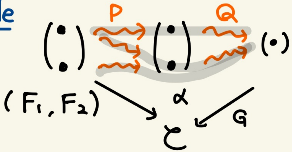
will play a key role.

They represent natural families of functions

$$\left(\alpha_{i,j} : P(i,j) \longrightarrow \mathcal{C}(Fi, Hj)\right)_{i,j}$$

$$\left(\beta_{i,j,k} : P(i,j) \times Q(j,k) \longrightarrow \mathcal{C}(Fi, Gk)\right)_{i,j,k}$$

Example



The data above amounts to

$$\begin{tikzcd}
F_1 \arrow[dr, bend left, "f"] \arrow[dr, bend right, "g"'] & \\
& G \\
F_2 \arrow[ur, bend right, "h"'] &
\end{tikzcd}$$

in $\mathcal{C}$

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Anticolimits via profunctors: balances

We take the example of regular epimorphisms as a model case,

and will solve the inverse problem of $X \underset{g}{\overset{f}{\rightrightarrows}} A \xmapsto{\text{colimit}} A \xrightarrow{e} Z$

Fixed data

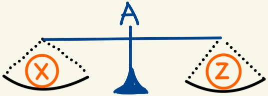
Def [Street. '80]

A gamut is a cell of the form

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "N", "\mid" marking] & \\ \mathcal{J} \arrow[ur, "M", "\mid" marking] \arrow[rr, bend right, "P"', "\mid" marking] & \mu & \mathcal{K} \end{tikzcd}$$

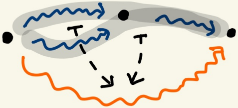
Def A balance on $\mathcal{C}$ consists of

a gamut μ and $\begin{tikzcd} \mathcal{I} \arrow[d, "A"] \\ \mathcal{C} \end{tikzcd}$ (fulcrum)



Ex $\mu_{coeq} :=$

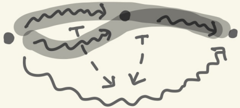
$$\begin{tikzcd} (\bullet) \arrow[r, "2", "\mid" marking] \arrow[rr, bend right, "1"', "\mid" marking, ""{name=B}] & (\bullet) \arrow[r, "1", "\mid" marking] \arrow[d, Rightarrow, shorten >=8pt] & (\bullet) \\ & {} & \end{tikzcd}$$



Ex A fulcrum for μ_{coeq} is an object $A \in \mathcal{C}$.

Anticolimits via profunctors: diagrams

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "N", "|" marking] & \\ \mathcal{J} \arrow[ur, "M", "|" marking] \arrow[rr, bend right, "P"', "|" marking] & \mu & \mathcal{K} \end{tikzcd}$$



Def (μ, A) : balance

A left diagram is a pair (X, ξ).

$$\begin{tikzcd} \mathcal{J} \arrow[r, "M", "|" marking] \arrow[dr, bend right, "X"'] & \mathcal{I} \arrow[d, "A"] \\ \xi & \mathcal{C} \end{tikzcd}$$

A right diagram is a pair (Z, ζ)

$$\begin{tikzcd} \mathcal{I} \arrow[r, "N", "|" marking] \arrow[d, "A"'] & \mathcal{K} \arrow[dl, bend left, "Z"] \\ \mathcal{C} & \zeta \end{tikzcd}$$

Ex For (μ_{coeq}, A),

$$\begin{pmatrix} \text{Left} \\ \text{diagrams} \end{pmatrix} = \begin{pmatrix} \text{Parallel arrows} \\ X \overset{f}{\underset{g}{\rightrightarrows}} A \quad \text{into } A \end{pmatrix}$$

$$\begin{pmatrix} \text{Right} \\ \text{diagrams} \end{pmatrix} = \begin{pmatrix} \text{Arrows} \\ A \xrightarrow{e} Z \quad \text{from } A \end{pmatrix}$$

The left/right diagrams constitute categories **LD**(μ, A) and **RD**(μ, A).

$(X, \xi) \xrightarrow{\sigma} (X', \xi')$ in **LD**(μ, A) $\quad \Big\|$

$$\begin{tikzcd} \mathcal{J} \arrow[d, bend right=50, "X"'] \arrow[d, bend left=50, "X'"] \arrow[d, phantom, "\sigma"] \\ \mathcal{C} \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} \mathcal{J} \arrow[r, "M", "|" marking] \arrow[dr, "X'"'] \arrow[dr, bend right=60, "X"'] \arrow[dr, phantom, bend right=30, "\sigma"] & \mathcal{I} \arrow[d, "A"] \arrow[dl, phantom, "\xi'"] \\ & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{J} \arrow[r, "M", "|" marking] \arrow[dr, "X"'] & \mathcal{I} \arrow[d, "A"] \arrow[dl, phantom, "\xi"] \\ & \mathcal{C} \end{tikzcd}$$

Anticolimits via profunctors: bicones

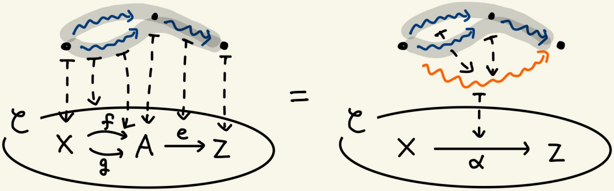
Def A bicone of (μ, A) is a triple of

- a left diagram $\xi = (X, \xi)$
- a right diagram $\zeta = (Z, \zeta)$, and
- a cell α such that

$$\begin{tikzcd} & & \mathcal{I} \arrow[rrd, "N", "\shortmid"{marking}] \arrow[dd] & & \\ \mathcal{J} \arrow[rru, "M", "\shortmid"{marking}] \arrow[rrd, "X"'] & \xi & & \zeta & \mathcal{K} \arrow[lld, "Z"] \\ & & \mathcal{C} & & \end{tikzcd} = \begin{tikzcd} & \mathcal{I} \arrow[rd, "N", "\shortmid"{marking}] & \\ \mathcal{J} \arrow[ru, "M", "\shortmid"{marking}] \arrow[rr, "\mu", "\shortmid"{marking}] \arrow[rdd, "X"'] & & \mathcal{K} \arrow[ldd, "Z"] \\ & \alpha & \\ & \mathcal{C} & \end{tikzcd} .$$

α is a bicone over ξ and ζ.

Ex For (μ_{coeq}, A),



α is unique if it exists.

Bicone-profunctor/General results on profunctors (1)

Prop Bicone : $LD(\mu, A)^{op} \times RD(\mu, A) \to Set$ is a functor,

$$(x, \xi) \quad , \quad (z, \zeta) \mapsto \left\{ \text{bicones over } \begin{matrix} (x,\xi) \\ (z,\zeta) \end{matrix} \right\}$$

which gives a profunctor Bicone : $LD(\mu, A) \nrightarrow RD(\mu, A)$.

Lem (Two-sided Grothendieck construction)

For a profunctor $T : A \nrightarrow B$, the category $\mathcal{T}$ defined by

- objects : $(a \in A,\ b \in B,\ t \in T(a,b))$
- arrows $\begin{matrix}(a,b,t) \\ \downarrow \\ (a',b',t')\end{matrix} : \left(\begin{matrix} a \\ f \downarrow \\ a' \end{matrix} \ , \ \begin{matrix} b \\ \downarrow g \\ b' \end{matrix} \right)$ s.t. $f \triangleleft t' = t \triangleright g$

induces the two-sided discrete fibration

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "L"'] \arrow[dr, "R"] & \\ A & & B \end{tikzcd}.$$

For BiCone, we write $\mathbf{Bico}(\mu, A)$ for this category $\mathcal{T}$.

$$\begin{tikzcd} & \mathbf{Bico}(\mu,A) \arrow[dl] \arrow[dr] & \\ LD(\mu,A) & & RD(\mu,A) \end{tikzcd}$$

Notation
- ${}_a\mathcal{T}$: the fiber of $\mathcal{T} \xrightarrow{L} A$ at $a \in A$.
- $\mathcal{T}_b$: the fiber of $\mathcal{T} \xrightarrow{R} B$ at $b \in B$.

Anticolimits via profunctors: limit / colimit bicone

Def A colimit bicone of $\xi \in \mathrm{LD}(\mu, A)$ is an initial objects in ${}_{\xi}\mathrm{Bico}(\mu, A)$.

A limit bicone of $\zeta \in \mathrm{RD}(\mu, A)$ is a terminal objects in $\mathrm{Bico}(\mu, A)_{\zeta}$.

$\begin{pmatrix}\text{Left} \\ \text{diagrams}\end{pmatrix} = \begin{pmatrix}\text{Parallel arrows} \\ X \overset{f}{\underset{g}{\rightrightarrows}} A \quad \text{into } A\end{pmatrix}$

$\begin{pmatrix}\text{Right} \\ \text{diagrams}\end{pmatrix} = \begin{pmatrix}\text{Arrows} \\ A \xrightarrow{e} Z \quad \text{from } A\end{pmatrix}$

for (μ_{coeq}, A),

Ex The case of (μ_{coeq}, A).

- For $\xi := \left(X \overset{f}{\underset{g}{\rightrightarrows}} A\right)$, ${}_{\xi}\mathrm{Bico} \cong \left\{ \begin{matrix} A \\ \downarrow k \\ Y \end{matrix} \;\middle|\; \underbrace{X \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{k} Y}_{:\text{cofork}} \right\} \overset{\text{f.f.}}{\hookrightarrow} A/\mathscr{C}$.

A colimit bicone is a coequalizer diagram.

- For $\zeta := \left(A \xrightarrow{e} Z\right)$, $\mathrm{Bico}_{\zeta} \cong \left\{ A \xleftarrow{p} W \xrightarrow{q} A \;\middle|\; \underbrace{W \overset{p}{\underset{q}{\rightrightarrows}} A \xrightarrow{e} Z}_{:\text{cofork}} \right\} \overset{\text{f.f.}}{\hookrightarrow} {}_{A}\mathrm{Span}(\mathscr{C})_{A}$.

A limit bicone is a kernel pair diagram.

General results on profunctors (2)

Lem For a two-sided discrete fibration $$\begin{tikzcd} & T \arrow[dl, "L"'] \arrow[dr, "R"] & \\ A & & B \end{tikzcd}$$,

(i) L has a left adjoint iff ${}_aT$ has an initial for all $a \in A$.

(ii) R has a right adjoint iff T_b has a terminal for all $b \in B$.

The values of these adjoints are given by the initials/terminals.

The resulting adjoints are fully-faithful.

Assuming the existence of limit/colimit bicones, we obtain:

$$\boxed{\begin{tikzcd} \mathrm{LD}(\mu, A) \arrow[r, shift left=1.5, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathrm{Bico}(\mu, A) \arrow[l, shift left=1.5, "L"] \arrow[r, shift left=1.5, "R"] \arrow[r, phantom, "\perp"] & \mathrm{RD}(\mu, A) \arrow[l, shift left=1.5, "\mathrm{Lim}"] \end{tikzcd}.}$$

Ex For $(\mu_{\mathrm{coeq.}}, A)$, this adjunction is:

$$\begin{tikzcd} {}_A\mathrm{Span}(\mathcal{C})_A \arrow[r, shift left=1.5, "\mathrm{coeq.}"] \arrow[r, phantom, "\perp"] & A/\mathcal{C} \arrow[l, shift left=1.5, "\mathrm{kerpair.}"] \end{tikzcd}.$$

Remark Most of the results can be developed without assuming (co) limits; we could use relative adjoints on both sides.

General results on profunctors (3)

$$\boxed{\begin{tikzcd} LD(\mu,A) \arrow[r, hook, shift left, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & Bico(\mu,A) \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & RD(\mu,A). \arrow[l, hook', shift left, "\mathrm{Lim}"] \end{tikzcd}}$$

Remark Idempotency of adjunctions are described in various ways.

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "G"] \end{tikzcd}$$ is idempotent $\Leftrightarrow$ $F\eta$: iso $\Leftrightarrow$ $\mathrm{Fix}(FG) = \mathrm{Im}(F) \subseteq B$

$:= \{ b \mid \varepsilon_b : \mathrm{iso} \}$

$\Leftrightarrow$ …

Prop If $T : A \nrightarrow B$ is a propositional profunctor and induces

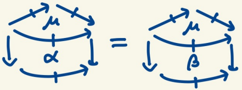
$:\Leftrightarrow \# T(a,b) \leqslant 1.$

the adjunction $$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \Upsilon \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left] \end{tikzcd}$$, then this is idempotent.

Ex If the gamut μ is epimorphic w.r.t. vertical composition,

Bicone is propositional.

μ_{coeq} is of this kind.



$\Rightarrow \alpha = \beta.$

Main result : anticolimit and effectiveness

$$\begin{tikzcd} LD(\mu, A) \arrow[r, shift left, "Colim"] \arrow[r, phantom, "\perp"] & Bico(\mu, A) \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & RD(\mu, A) \arrow[l, shift left, "Lim"] \end{tikzcd}$$

Thm Let (μ, A) be a balance.

If Bicone of this is propositional,

then the following are equivalent for $\zeta \in RD(\mu, A)$:

(i) $\zeta \cong R \circ Colim(\xi)$ for some $\xi \in LD(\mu, A)$.

$:\Longleftrightarrow$ ζ has an anticolimit.

(ii) $\zeta \cong R \circ Colim \circ L \circ Lim(\zeta)$ (canonically)

ζ is "the colimit of the limit" of itself.

Ex For (μ_{coeq}, A),

(i) e : regular epi

(ii) e : coeq. of its kernel pair

Cor The adjunction reduces to an equivalence :

$$\left\{ \xi \in LD \;\middle|\; \begin{matrix} \xi \text{ has} \\ \text{an antilimit} \end{matrix} \right\} \simeq \left\{ \zeta \in RD \;\middle|\; \begin{matrix} \zeta \text{ has} \\ \text{an anticolimit} \end{matrix} \right\}.$$

Structure

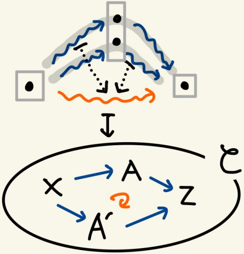
1. Elements of anticolimit.
2. Formal category theory via (virtual) double categories.
3. Formal theory of anticolimits : general theory
4. Formal theory of anticolimits : application

Captured Examples (1)

① Set-enriched case

(i) $\mu_{\mathrm{pbpo}} :=$

$$\begin{tikzcd}
& 2 \arrow[dr, "1", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow !"] & \\
1 \arrow[ur, "1", "\shortmid"{marking}] \arrow[rr, "1"', "\shortmid"{marking}] & \, & 1
\end{tikzcd}$$



The resulting adj. is

$$\begin{tikzcd}
{}_A\mathrm{Span}(\mathcal{C})_{A'} \arrow[d, shift right=2, "\text{p.o.}"] \\
{}_A\mathrm{Cospan}(\mathcal{C})_{A'} \arrow[u, shift right=2, "\text{p.b.}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

(ii) S : set

$\mu_{\mathrm{mk},S} :=$

$$\begin{tikzcd}
& S \arrow[dr, "1", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow !"] & \\
S \times S \arrow[ur, "M", "\shortmid"{marking}] \arrow[rr, "1"', "\shortmid"{marking}] & \, & 1
\end{tikzcd}$$

where M only has $(s,s') \overset{l_{s,s'}}{\rightsquigarrow} s$, $(s,s') \underset{r_{s,s'}}{\rightsquigarrow} s'$ $(\forall s,s')$

$(\zeta_s : A_s \to Z)_{s \in S}$ has an anticolimit

iff it is a colimit of $\left(\begin{tikzcd}[row sep=tiny] & A_s \\ A_s \times_Z A_{s'} \arrow[ur] \arrow[dr] & \\ & A_{s'} \end{tikzcd} \right)_{s,s'}$.

② Ab-enriched case

$\mu_{\mathrm{ck}} :=$

$$\begin{tikzcd}
& \Delta 1 \arrow[dr, "\mathbb{Z}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow !"] & \\
\Delta 1 \arrow[ur, "\mathbb{Z}", "\shortmid"{marking}] \arrow[rr, "0"', "\shortmid"{marking}] & \, & \Delta 1
\end{tikzcd}$$

The resulting adjunction is

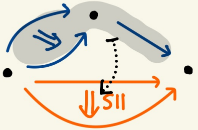
$$\begin{tikzcd}
\mathcal{C}/A \arrow[r, shift left=2, "\mathrm{Coker}"] \arrow[r, phantom, "\perp"] & A/\mathcal{C} \arrow[l, shift left=2, "\mathrm{Ker}"]
\end{tikzcd}.$$

Similar for $(\mathrm{Set}_*, \wedge)$-cats.

Captured Examples (2)

③ Cat-enriched case

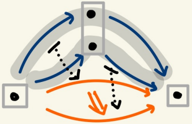
(i)



$$\begin{tikzcd} \left\{ X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & A \right\} \arrow[r, shift left=2, "\text{``Coinvt''}"] \arrow[r, phantom, "\perp"] & A/\mathcal{K}_0 \arrow[l, shift left=2] \end{tikzcd}$$

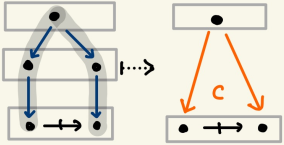
$$\begin{tikzcd} \mathrm{Lim}(\xi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z^{(\bullet\cong\bullet)} \arrow[d] & & A \arrow[d, "\xi"] \arrow[ll, mapsto, shorten >=2em] \\ A^{(\bullet\to\bullet)} \arrow[r, "\xi^{\bullet\to\bullet}"'] & Z^{(\bullet\to\bullet)} & & Z \end{tikzcd}$$

(ii) (⚠ Non-propositional)



$$\begin{tikzcd} {}_A\mathrm{Span}(\mathcal{K})_B \arrow[r, shift left=2, "\text{Cocomma}"] \arrow[r, phantom, "\perp"] & {}_A\mathrm{Cospan}(\mathcal{K})_B \arrow[l, shift left=2, "\text{Comma}"] \end{tikzcd}$$

④ Double categories (propositional if $\mathbb{D}$: flat)



$$\begin{tikzcd} {}_A\mathrm{Span}(\mathbb{D}_0)_B \arrow[r, shift left=2, "\text{Ext}"] \arrow[r, phantom, "\perp"] \arrow[dr, shift left=1] & \left\{ \begin{array}{ccc} A & & B \\ \downarrow & & \downarrow \\ \bullet & \xrightarrow{\;\;\mid\;\;} & \bullet \end{array} \text{ in } \mathbb{D} \right\} \arrow[l, shift left=2, "\text{Tab}"] \\ & \mathbb{D}(A,B) \arrow[u, hook] \arrow[ul, shift left=1, "\perp"] \end{tikzcd}$$

Future application

(i) Formal theory of homology ?

For a gamut of the form

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "M", "|" marking] & \\ \mathcal{I} \arrow[ur, "M", "|" marking] \arrow[rr, "P"', "|" marking] \arrow[rr, phantom, bend left=20, "\mu"] & & \mathcal{I} \end{tikzcd}$$

(vertically epimorphic)

we can define chain complex as

$$\begin{tikzcd} \cdots \arrow[r, "|" marking] & \mathcal{I} \arrow[r, "M", "|" marking] \arrow[ddr, bend right, "C_2"'] & \mathcal{I} \arrow[r, "M", "|" marking] \arrow[dd, "C_1"] & \mathcal{I} \arrow[r, "M", "|" marking] \arrow[ddl, "C_0"'] & \mathcal{I} \arrow[r, "|" marking] \arrow[ddll, bend left, "C_{-1}"] & \cdots \\ & \varphi_{21} & \varphi_{10} & \varphi_{0,-1} & & \\ & & \mathcal{C} & & & \end{tikzcd}$$

s.t. the composites of two adjacent φ's factor through μ.

Ex

$$\begin{tikzcd} & \Delta 1 \arrow[dr, "\mathbb{Z}", "|" marking] & \\ \Delta 1 \arrow[ur, "\mathbb{Z}", "|" marking] \arrow[rr, "0"', "|" marking] \arrow[rr, phantom, bend left=20, "\Downarrow !"] & & \Delta 1 \end{tikzcd}$$

in Ab-Prof

leads to a usual one.

(ii) Formal theory of regularity ?

Bourke and Garner developed theory of 2-dimensional regularity based on kernel-quotient systems.

Def [BG14] A kernel-quotient system is an f.f. embedding $J : \underset{(0 \to 1)}{\overset{}{2}} \hookrightarrow F$

$$2 = (0 \to 1)$$

Let $K := F \setminus \{J1\} \overset{\text{f.f.}}{\hookrightarrow} F$

With this, they construct an adjunction

$$\begin{tikzcd} {[K, \mathcal{C}]} \arrow[r, shift left, "\text{quot}"] \arrow[r, phantom, "\perp"] & {[2, \mathcal{C}]} \arrow[l, shift left, "\text{ker}"] \end{tikzcd}$$

We can recover this by taking μ as

$$\begin{tikzcd} & 1 \arrow[dr, "I", "|" marking] & \\ F \setminus \mathrm{Im} J \arrow[ur, "{F(-, J0)}", "|" marking] \arrow[rr, "{F(-, J1)}"', "|" marking] \arrow[rr, phantom, bend left=20, "\Downarrow \cdots (J0 \to J1)_*"] & & 1 \end{tikzcd}$$

Future application

(i) Formal theory of homology ?

For a gamut of the form

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "M", "|" marking] & \\ \mathcal{I} \arrow[ur, "M", "|" marking] \arrow[rr, "P"', "|" marking] & {\mu} & \mathcal{I} \end{tikzcd}$$

(vertically epimorphic)

we can define chain complex as

$$\begin{tikzcd} \cdots \arrow[r, "|" marking] & \mathcal{I} \arrow[rr, "M", "|" marking] \arrow[ddrrr, bend right, "C_2"'] & & \mathcal{I} \arrow[rr, "M", "|" marking] \arrow[ddr, "C_1"] & & \mathcal{I} \arrow[rr, "M", "|" marking] \arrow[ddl, "C_0"'] & & \mathcal{I} \arrow[r, "|" marking] \arrow[ddlll, bend left, "C_{-1}"] & \cdots \\ & & \varphi_{21} & & \varphi_{10} & & \varphi_{0,-1} & & \\ & & & & \mathcal{C} & & & & \end{tikzcd}$$

s.t. the composites of two adjacent φ's factor through μ.

Ex

$$\begin{tikzcd} & \Delta 1 \arrow[dr, "\mathbb{Z}", "|" marking] \arrow[d, phantom, "\Downarrow\, !"] & \\ \Delta 1 \arrow[ur, "\mathbb{Z}", "|" marking] \arrow[rr, "0"', "|" marking] & {} & \Delta 1 \end{tikzcd}$$

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Bourke and Garner developed theory of 2-dimensional regularity based on kernel-quotient systems.

Def [BG14] A kernel-quotient system is an f.f. embedding $J : \underset{\overset{\shortparallel}{(0\to 1)}}{2} \hookrightarrow F$

Let $K := F \backslash \{J1\} \xhookrightarrow{\text{f.f.}} F$

With this, they construct an adjunction

$$\begin{tikzcd} {[K, \mathcal{C}]} \arrow[r, shift left=1.5ex, "\text{quot}"] \arrow[r, phantom, "\perp"] & {[2, \mathcal{C}]} \arrow[l, shift left=1.5ex, "\text{ker}"] \end{tikzcd}$$

We can recover this by taking μ as

$$\begin{tikzcd} & 1 \arrow[dr, "I", "|" marking] \arrow[d, phantom, "\Downarrow\ (J0\to J1)_*"] & \\ F\backslash \mathrm{Im}J \arrow[ur, "{F(-,J0)}", "|" marking] \arrow[rr, "{F(-,J1)}"', "|" marking] & {} & 1 \end{tikzcd}$$

Final slide

Summary

- Anticolimit is the inverse problem of colimits.
  Solutions are occasionally given "effectively".
- We provided a general theory of anticolimits in virtual equipments.
- We constructed the (possibly relative) adjoint of limit and colimit from a data of diagram shape (= gamut).

$$\begin{tikzcd}
LD(\mu, A) \arrow[r, shift left, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathrm{Bico}(\mu, A) \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & RD(\mu, A). \arrow[l, shift left, "\mathrm{Lim}"]
\end{tikzcd}$$

What I couldn't include

- Pointwiseness of limits / colimits
- The cases of relative adjoints

What I want to look into

- Preservation / Reflection / Stability
- The two applications
  in the previous slide.

References

[TV24] Callin Tataru and Jamie Vicary. The theory and applications of anticolimits. 2024. preprint on arXiv.

[BG14] John Bourke and Richard Garner. Two-dimensional regularity and exactness. 2014. Journal of Pure and Applied Algebra.

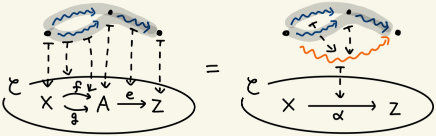
[St80] Ross Street. Fibrations in bicategories. 1980, Cahier de Topologie et Géométrie Différentielle.

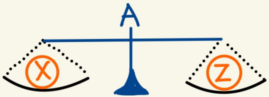
[nLab+] nLab page. Generalized kernels

Thank you!

hnasu@kurims.kyoto-u.ac.jp
hayatonasu.github.io

Please let me know if you hit upon any example of anticolimit!





$$\begin{tikzcd} LD(\mu, A) \arrow[r, shift left=1.5, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathrm{Bico}(\mu, A) \arrow[l, shift left=1.5, "L"] \arrow[r, shift left=1.5, "R"] \arrow[r, phantom, "\perp"] & RD(\mu, A). \arrow[l, shift left=1.5, "\mathrm{Lim}"] \end{tikzcd}$$