On the decomposition of a strong epimorphism into regular morphisms

Hayato Nasu

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jww. Yuto Kawase (Kyoto Univ.)

What is the most fundamental theorem in (abstract) algebra?

A. The fundamental theorem of homomorphisms!

$$A/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$$

- For groups $\varphi : G \to H \rightsquigarrow G/\underbrace{\mathrm{Ker}(\varphi)}_{\text{normal subgroup}} \cong \mathrm{Im}(\varphi)$

- For rings $\varphi : R \to S \rightsquigarrow R/\underbrace{\mathrm{Ker}(\varphi)}_{\text{ideal}} \cong \mathrm{Im}(\varphi)$

- For modules $\varphi : M \to N \rightsquigarrow M/\underbrace{\mathrm{Ker}(\varphi)}_{\text{submodule}} \cong \mathrm{Im}(\varphi)$

The common idea of $-/\mathrm{Ker}(\varphi)$:

identifying two elements that go to the same element

What about categories? No!

Counter example :

$$\boxed{\mathcal{C}} \qquad\qquad\qquad\qquad\qquad\qquad \boxed{\mathcal{D}}$$

$$\left(\begin{tikzcd} & B & B' \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rrr, "h"'] & & & C \end{tikzcd}\right) \xrightarrow{\ \Phi\ } \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right)$$

$$\mathcal{C}/\mathrm{Ker}(\Phi) = \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\neq"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right)$$

$$\not\cong$$

This is undefined!

What should be identified?

$B = B'$, $\cancel{g \circ f = h}$

Counter example :

$$\boxed{\mathcal{C}} \qquad \left(\begin{tikzcd} & B & B' \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rrr, "h"'] & & & C \end{tikzcd}\right) \xrightarrow{\ \Phi\ } \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right) \qquad \boxed{\mathcal{D}}$$

$$\mathcal{C}/\mathrm{Ker}(\Phi) = \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\neq"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right) \xrightarrow{\ \Phi'\ } \mathcal{D}$$

$$\Rightarrow \left(\mathcal{C}/\mathrm{Ker}(\Phi)\right)/\mathrm{Ker}(\Phi') \cong \mathrm{Im}(\Phi)$$

1. The decomposition numbers

2. For models of generalized algebraic theories

3. For models of partial Horn theories

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Quotient by kernels

A **regular epimorphism** is a coequalizer of some parallel pair.

Proposition. $\mathcal{C}$: a category w/ pullbacks

$f : A \to B$ is a regular epimorphism

$\Longleftrightarrow$ $\mathrm{Ker}(f) \underset{p_2}{\overset{p_1}{\rightrightarrows}} A \xrightarrow{f} B$: coequalizer, where

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Starting from $A \xrightarrow{f} B$, we obtain a sequence of reg. epis

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[drrrr, bend right, "f"'] & A_1 \arrow[r, two heads] \arrow[drrr, bend right, "f_1"] & A_2 \arrow[r, two heads] \arrow[drr, bend right, "f_2"] & A_3 \cdots \arrow[dr, bend right, "\cdots"] & \\ &&&& B \end{tikzcd}$$

transfinitely

The image of a mophism

$f : A \to B$ is a strong epimorphism if
$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & X \arrow[d, tail, "m"] \\ B \arrow[r, "\ell"'] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd} \quad \left(\begin{array}{l} \forall m : \text{mono} \\ \forall k \quad \forall \ell \end{array} \right)$$

When a category has all pullbacks

$\Longleftrightarrow$ $A \xrightarrow{f} B$ factors through no proper $B' \rightarrowtail B$

If a morphism $f : A \to B$ can be presented as
$$\underset{\text{str.epi.}}{A \xrightarrow{k} I} \underset{\text{mono}}{\overset{m}{\rightarrowtail}} B,$$
then the factorization is unique up to isomorphism.

We call I the image of f.

reg.epi.

str.epi.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr] & B \\
A/\mathrm{Ker}(f) \arrow[r, dashed, "\exists !"'] & \mathrm{Im}(f) \arrow[u, tail]
\end{tikzcd}$$

When {reg.epis} = {str. epis}, this morphism is an iso.

Proposition. [Kelly '69, Gabriel, Ulmer '71]

In a locally presentable category, the transfinite sequence

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[drrrr, bend right, "f"'] & A_1 \arrow[r, two heads] \arrow[drrr, bend right, "f_1"] & A_2 \arrow[r, two heads] \arrow[drr, bend right, "f_2"] & A_3 \cdots & \cdots \arrow[d] \\
& & & & B
\end{tikzcd}$$

stabilizes at some λ, and gives

$$\begin{tikzcd}
A \arrow[r, two heads, "\text{str.epi.}"'] & A_\lambda \arrow[r, tail, "\text{mono}"'] & B
\end{tikzcd}$$

$A_\lambda = \mathrm{Im}(f)$,

The decomposition number

Definition. [Gabriel, Ulmer '71, Börger '91, KN.]

The regular (canonical) decomposition number $\delta(f)$ of $A \xrightarrow{f} B$ is the smallest α at which the seq. stabilizes ($A_{\alpha,\alpha+1}$: iso).

$$\begin{tikzcd}
A \arrow[r, two heads, "A_{0,1}"] \arrow[drrrr, bend right, "f"'] & A_1 \arrow[r, two heads, "A_{1,2}"] \arrow[drrr, bend right, "f_1"'] & A_2 \arrow[r, two heads, "A_{2,3}"] \arrow[drr, bend right, "f_2"'] & A_3 \cdots & \\
 & & & & B
\end{tikzcd}$$

The global regular (canonical) decomposition number $\delta(\mathcal{C})$ of a category $\mathcal{C}$ is the smallest γ s.t. $\delta(f) < \gamma$ for any f in $\mathcal{C}$.

Proposition. [Gabriel, Ulmer '71, K.N.]

$\mathcal{C}$: locally λ-presentable $\Rightarrow$ $\delta(\mathcal{C}) \leq \lambda + 1$.

Examples

- Grp, Ring, Module$_R$: $\delta = 2$

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "f"] \arrow[d, two heads] & B \\ & A/\mathrm{Ker}(f) \cong \mathrm{Im}(f) \arrow[ur, hook] & \end{tikzcd}$$

- Any regular category : $\delta \leq 2$
- $\delta(\mathbf{Pos}) = 2$
- $\delta(\mathbf{Field}) = 1$ $(\because \ K \cong \mathrm{Im}(f))$
- [Bednarczyk, et.al. '99] $\delta(\mathbf{Cat}) = 3$

Any strong epi is the composite of two regular epis.

$\left(\Phi : A \to B \text{ is a str. epi} \Leftrightarrow \text{surj. on obj.} + \begin{array}{l} \forall g \text{ in } B \ \ \exists f_1, \dots, f_n \\ \quad g = \Phi(f_n) \circ \cdots \circ \Phi(f_1) \end{array} \right)$

$$\delta(n\text{-}\mathrm{Cat}) = \ ?$$

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$$\left(\begin{tikzcd}
 & B & B' \arrow[dr, "h"] & \\
A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\varphi"] \arrow[rrr, bend left, "k"] \arrow[rrr, "\ell"] \arrow[rrr, bend right, "m"'] \arrow[rrr, phantom, shift right=3, "\Downarrow\tau"] \arrow[rrr, phantom, shift left=3, "\Downarrow\chi"] & & & C
\end{tikzcd}\right)
\xrightarrow{\ \Phi\ }
\left(\begin{tikzcd}
 & B \arrow[dr, bend left, "h"] \arrow[d, phantom, "\Downarrow\chi"] & \\
A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\varphi"] \arrow[rr, bend right, "m"'] & \, & C
\end{tikzcd}\right)$$

$B' \mapsto B$

$k \mapsto h \circ g$

$\ell \mapsto h \circ f$

$\tau \mapsto \chi \circ (h\varphi)$

$$\Big\downarrow\ B = B' \qquad\qquad\qquad\qquad \Big\uparrow\ \tau = \chi \circ (h\varphi)$$

$$\left(\begin{tikzcd}
 & B \arrow[dr, bend left, "h"] & \\
A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\varphi"] \arrow[rr, bend left, "k"] \arrow[rr, "\ell"] \arrow[rr, bend right, "m"'] \arrow[rr, phantom, shift right=3, "\Downarrow\tau"] \arrow[rr, phantom, shift left=3, "\Downarrow\chi"] & & C
\end{tikzcd}\right)
\longrightarrow\!\!\!\!\rightarrow
\left(\begin{tikzcd}
 & B \arrow[dr, bend left, "h"] \arrow[d, phantom, "\tau\Downarrow\ \Downarrow\chi"] & \\
A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\varphi"] \arrow[rr, bend right, "m"'] & \, & C
\end{tikzcd}\right)$$

$h \circ f = \ell$

$h \circ g = k$

$$\delta(\Phi) = 3 \ \rightsquigarrow\ \delta(2\text{-}\mathrm{Cat}) \geq 4 .$$

Is this $=$?

1. The decomposition numbers

2. For models of generalized algebraic theories

3. For models of partial Horn theories

Theories & models

$$\mathrm{Grp} \simeq \left\{ \begin{pmatrix} X,\ X^2 \xrightarrow{\cdot} X, \\ 1 \xrightarrow{e} X,\ X \xrightarrow{(-)^{-1}} X \end{pmatrix} \middle|\ \begin{matrix}\text{group} \\ \text{axioms}\end{matrix} \right\} \simeq \mathrm{FPCAT}(\underline{\mathbb{T}_{\mathrm{Grp}}}, \mathrm{Set})$$

$$\mathrm{FPCAT}(\mathbb{T}_{\mathrm{Grp}}, \mathrm{Set}) = \mathrm{Mod}(\mathbb{T}_{\mathrm{Grp}})$$

The equational theory of groups

The Lawvere theory of groups

Proposition. $\mathbb{T}$: Lawvere theory $\Rightarrow$ $\mathrm{Mod}(\mathbb{T})$: regular

Theories $\mathbb{T}$	$\longrightarrow$	Categories of models $\mathrm{Mod}(\mathbb{T})$
satisfies P_n	$\Longrightarrow$	$\delta \leq n$

Equational theory $\approx$ Lawvere theory $\xrightarrow{\mathrm{Mod}\,=\,\mathrm{FPCAT}(-,\mathrm{Set})}$ Categories of algebras

(operations + equational axioms)

Cartesian theory [Johnstone '02] $\approx$ Categories with finite limits $\xrightarrow{\mathrm{Mod}}$

Essentially algebraic theory [Freyd '72] $\xrightarrow{\mathrm{Mod}}$ Locally finitely presentable categories

partial Horn logic [Palmgren, Vickers '07] $\xrightarrow{\mathrm{Mod}}$

generalized algebraic theory [Cartmell '78] $\approx$ clans [Joyal '17] $\xrightarrow{\mathrm{Clan}(-,\mathrm{Set})}$ Locally finitely presentable categories

Generalized algebraic theory

$\boxed{\text{(1-sorted) algebras}}$

$\rightsquigarrow$

$\boxed{\text{many-sorted algebras}}$

e.g. Module : the cat. of pairs (R, M)

ring, R-module

$$\mathrm{Module} = \mathrm{Mod}(\Pi_{\mathrm{Module}})$$

Π_{Module}

Sorts $\vdash R$, $\vdash M$

Operators $a : R,\ b : R \vdash a \circ b : R$

$a : R,\ x : M \vdash a \bullet x : M$

$\vdots$

Axioms $a : R,\ b : R,\ x : M$

$\vdash (a \circ b) \bullet x = a \bullet (b \bullet x) : M$

$\vdots$

Generalized algebraic theory

$$\boxed{\text{1-sorted algebras}}$$
$$\rotatebox{-90}{$\subsetneq$}$$
$$\boxed{\text{many-sorted algebras}}$$
$$\rotatebox{-90}{$\subsetneq$}$$
$$\boxed{\text{dependent-sorted algebras}}$$

Π_{Module}

Sorts $\quad \vdash R \;,\; \vdash M$

Operators $\quad a : R,\ b : R \vdash a \circ b : R$

$\qquad a : R,\ x : M \vdash a \bullet x : M$

$\vdots$

Axioms $\quad a : R,\ b : R,\ x : M$

$\qquad \vdash (a \circ b) \bullet x = a \bullet (b \bullet x) : M$

$\vdots$

e.g. $\mathrm{Cat} = \mathrm{Mod}(\Pi_{\mathrm{cat}})$

Π_{cat}

Sorts

$$\vdash \mathrm{Ob}$$

$$x : \mathrm{Ob},\ y : \mathrm{Ob} \vdash \mathrm{Mor}(x, y)$$

Operators

$$x : \mathrm{Ob} \vdash \mathrm{id}(x) : \mathrm{Mor}(x, x)$$

$$x, y, z : \mathrm{Ob},\ f : \mathrm{Mor}(x, y),\ g : \mathrm{Mor}(y, z)$$

$$\vdash \underbrace{\mathrm{comp}(x, y, z, f, g)}_{g \circ f} : \mathrm{Mor}(x, z)$$

Axioms

$$x : \mathrm{Ob},\ f : \mathrm{Mor}(x, y) \vdash f \circ \mathrm{id}(x) = f$$

$$: \mathrm{Mor}(x, y)$$

Proposition. [Cartmel '78, Adámek, Rosický '94]

$\mathcal{C}$: locally finitely presentable $\iff \exists\, \mathbb{T}$: GAT $\quad \mathcal{C} \simeq \mathrm{Mod}(\mathbb{T})$

Given a GAT $\mathbb{T}$, how can we know $\delta(\mathrm{Mod}(\mathbb{T}))$?

Observation.

$$\left(\begin{tikzcd} & B & B' \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rrr, "h"'] & & & C \end{tikzcd}\right) \xrightarrow{\Phi} \left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right)$$

$B = B'$

Identifying non-dependent terms.

$$\left(\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\neq"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & {} & C \end{tikzcd}\right)$$

$g \circ f = h$

Identifying terms of "dependency rank 1".

The dependency rank $\rightsquigarrow \delta$

Definition. In a GAT $\mathbb{T}$, we define the dependency rank $dr(A)$ for each well-formed type $\Gamma \vdash A$ by

$$dr(A) = \max\{dr(B) \mid B \text{ appear in } \Gamma\} + 1.$$

Example. $dr(Ob) = 1$, $dr(Mor(x,y)) = 2$

We call $\mathbb{T}$ a non-descending if

- $\Gamma \vdash f(\vec{x}) : B$: operator symbol
- $\Gamma \vdash t = s : B$: axiom

$\Longrightarrow \quad dr(B) \geq \max_{A \text{ in } \Gamma} dr(A)$

Theorem. [KN.] $\mathbb{T}$: non-descending, $\max_{A \text{ in } \mathbb{T}} dr(A) \leq n$

$\Rightarrow \forall g : M \to N$ in $Mod(\mathbb{T})$ $\delta(f) \leq n+1$, hence $\delta(Mod(\mathbb{T})) \leq n+2$

Examples

- $\mathbf{nCat} = \mathbf{Mod}(\mathbb{T}_{ncat})$

Sorts $\vdash \underline{0}$, $x, x' : \underline{0} \vdash \underline{1}(x, x')$, $x, x' : \underline{0}$, $y, y' : \underline{1}(x, x') \vdash \underline{2}(x, x', y, y')$ ...

Operators $x, x' : \underline{0}$, $y, y', y'' : \underline{1}(x, x')$, $z : \underline{2}(x, x', y, y')$, $z' : 2(x, x', y', y'')$

$\vdash \text{v-comp}(x, x', y, y', y'', z, z') : \underline{2}(x, x', y, y'')$ ...

$dr(\underline{n}) = n+1$, so $\delta(\mathbf{nCat}) \leq n+2$. In fact, $\delta(\mathbf{nCat}) = n+2$.

- $\delta(\mathbf{strMonCat}) = 3$.

- $\delta(\mathbf{strDblCat}) = 4$.

not non-descending!

- $\exists\, \mathbb{T} : \mathrm{GAT}$ with $\max\{dr(A)\} = 1$. $\delta(\mathbf{Mod}(\mathbb{T})) = \omega + 1$.

Strategy of the proof

$\max_{A \text{ in } \mathbb{T}} dr(A) = n$, $\mathbb{T}$: non-descending

$\Longrightarrow$ $\mathbb{T}_0 \longrightarrow \mathbb{T}_1 \longrightarrow \cdots \longrightarrow \mathbb{T}_n = \mathbb{T}$: a seq. of theories

with $\delta(\mathrm{Mod}(\mathbb{T}_{i+1})) \leq \delta(\mathrm{Mod}(\mathbb{T}_i)) + 1 \quad (\forall i)$

Example. $\mathbb{T}_\cdot \to \mathbb{T}_{\mathrm{Set}} \to \mathbb{T}_{\mathrm{Cat}} \to \mathbb{T}_{2\mathrm{Cat}} \to \cdots \to \mathbb{T}_{n\mathrm{Cat}}$

We derive a sequence of their syntactic categories

$$\mathcal{C}_{\mathbb{T}_0} \hookrightarrow \mathcal{C}_{\mathbb{T}_1} \hookrightarrow \cdots \hookrightarrow \mathcal{C}_{\mathbb{T}_n} = \mathcal{C}_{\mathbb{T}},$$

so the problem reduces to find a nice property of these functors.

The syntactic categories are seen as clans.

Proof method: Clans

Definition. A clan is a pair of

- a category $\mathcal{C}$, and
- a class of morphisms D

s.t.

- D : closed under comp.
  contains all isos.
- $\mathcal{C}$ has a terminal.
- $! : C \to 1 \in D \quad (\forall C)$.
- $\mathcal{C}$ has p.b.s along maps in D.
- $$\begin{tikzcd} P \arrow[r] \arrow[d, "D \ni"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_3 \arrow[d, "\in D"] \\ C_1 \arrow[r] & C_2 \end{tikzcd}.$$

Example. $\mathbb{T}$: GAT $\rightsquigarrow (\mathcal{C}_\mathbb{T}, D_\mathbb{T})$

objects : contexts

morphisms : "tuples of terms"

$D_\mathbb{T} := \{\text{projections } \Gamma.\Delta \to \Gamma\}$

$$\mathrm{Mod}(\mathbb{T}) = \mathrm{Clan}((\mathcal{C}_\mathbb{T}, D_\mathbb{T}), \mathbf{Set})$$

$$\| \quad \mathrm{Mod}(\mathcal{C}_\mathbb{T})$$

Definition.

A clan morphism $\Phi : (\mathcal{C}, D) \to (\mathcal{C}', D')$ is a functor $\Phi : \mathcal{C} \to \mathcal{C}'$ s.t.

- $\Phi(D) \subseteq D'$
- Φ preserves the limits in Def.

Simple clan morphisms

We call a clan morphism $\Phi : (\mathcal{C}, D) \to (\mathcal{C}', D')$ simple if D' is the "smallest clan str." on $\mathcal{C}'$ that includes

$$\{ d : c' \to \Phi(c) \text{ in } \mathcal{C}' \mid d \in D' \}.$$

Example. • $\mathcal{C}_{set} \hookrightarrow \mathcal{C}_{cat}$ is simple.

• $\mathcal{C}_{set} \hookrightarrow \mathcal{C}_{2cat}$ is not simple.

For the sequence $\Pi_0 \to \cdots \to \Pi_n$, $\mathcal{C}_{\Pi_i} \underset{\hookrightarrow}{\overset{\exists}{\underset{\perp}{\leftarrow}}} \mathcal{C}_{\Pi_{i+1}}$ is simple.

removing types with $dr > i$.

Proposition. $\mathcal{C}' \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$: rari in the 2-category of clans.

G : simple $\Rightarrow$ $\delta(\mathrm{Mod}(\mathcal{C})) \leq \delta(\mathrm{Mod}(\mathcal{C}')) + 1$

1. The decomposition numbers

2. For models of generalized algebraic theories

3. For models of partial Horn theories

partial Horn logic

Operators are allowed to be partial.

Axioms are allowed to be Horn clauses, not just equations.
e.g. $xy = xz \vdash y = z$

Example (categories)

$$\left(\begin{array}{l} Ob,\ Mor,\ s : Mor \to Ob,\ t : Mor \to Ob, \\ id : Ob \to Mor,\ -\circ- : Mor \times Mor \to Mor \end{array}\right) \text{ s.t. } \begin{array}{l} \vdash s(x)\downarrow, \\ \vdash id(t(x)) \circ x = x, \\ \vdots \end{array}$$

$\downarrow$ is defined

$\rightsquigarrow$ $\mathbb{T}_{Cat}$: partial Horn theory of categories,

$$\mathbf{Cat} \simeq \mathrm{Mod}(\mathbb{T}_{Cat})$$

Proposition. [Palmgren, Vickers '07, Kawase '23]

$\mathcal{C}$: locally finitely presentable $\Leftrightarrow$ $\exists\, \mathbb{T}$: partial Horn th. $\mathcal{C} \simeq \mathrm{Mod}(\mathbb{T})$

Upper bound of δ using PHL (Sketch)

Theorem [KN.] If $\mathbb{T}$ comes with "a gauge of length γ", for any $f : M \to N$ in $\mathbf{Mod}(\mathbb{T})$, $\delta(f) \leq \gamma$. Thus, $\delta(\mathbf{Mod}(\mathbb{T})) \leq \gamma + 1$.

Idea: Layering all terms based on definedness.

Example (a gauge $(\#, \sigma)$ for $\mathbb{T}_{cat}$)

$$\text{Terms} \xrightarrow{\#} \text{Ord}$$

$$\left.\begin{array}{c} x : \mathrm{Ob}, \\ s(x) : \mathrm{Mor}, \\ id(t(x)) : \mathrm{Mor} \end{array}\right\} \longmapsto 0$$

$$\left. y \circ x : \mathrm{Mor} \right\} \longmapsto 1$$

$$P_{fin}(\#^{-1}(0)^2)$$

$$\rotatebox{90}{$\in$}$$

$$\{(\sigma_1^k(\tau), \sigma_2^k(\tau))\} \longleftarrow\!\shortmid \tau$$

$\#(\tau : A) = 0$

$\iff$ τ is defined in $\mathbb{T}$

$\#(\tau : A) = 1$

$\iff \left\{ \tau\downarrow \leftrightarrow \bigwedge_k \sigma_1^k(\tau) = \sigma_2^k(\tau) \right.$ in $\mathbb{T}$

Some Comments

GAT approach vs PHL approach

- GAT gives a simpler way to determine the upper bound of δ for a limited case.

- PHL approach is more general (even for locally λ-presentable categories), but finding a gauge is not a simple task.

Omitted topics ([KN.])

- The regular decomposition gives the most efficient way to present a str. epi as a composite of reg. epis

$$\begin{array}{c}\xrightarrow{\quad\quad f \quad\quad}\\ \shortparallel \\ \underbrace{\xrightarrow{f_1}\!\!\!\!\rightarrow \xrightarrow{f_2}\!\!\!\!\rightarrow \cdots \xrightarrow{f_n}\!\!\!\!\rightarrow}_{\text{reg. epis}}\end{array} \implies \delta(f) \leq n$$

- Generalization of the class of regular epimorphisms (localization functors in **Cat**, etc.)

Summary

- The fundamental theorem of homomorphisms states that strong epis $\Leftrightarrow$ regular epis in the category in question.
- They do not coincide in general, but in l.p. categories. strong epis $\Leftrightarrow$ transfinite composites of regular epis.
- The number of regular epis needed (= decomposition number) can have an upper bound determined by GAT/PHL presentation.

  GAT The dependency rank + 2 (when non-descending)

  PHL The length of a gauge + 1

References

[AHT89] Jií Adámek, Horst Herrlich, and Walter Tholen. Monadic decompositions. *J. Pure Appl. Algebra*, 59(2):111–123, 1989.

[ARV11] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic theories*, volume 184 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011. A categorical introduction to general algebra, With a foreword by F. W. Lawvere.

[B9̈1] Reinhard Börger. Making factorizations compositive. *Comment. Math. Univ. Carolin.*, 32(4):749–759, 1991.

[BBP99] Marek A. Bednarczyk, Andrzej M. Borzyszkowski, and Wieslaw Pawlowski. Generalized congruences—epimorphisms in *Cat*. *Theory Appl. Categ.*, 5:No. 11, 266–280, 1999.

[Car86] John Cartmell. Generalised algebraic theories and contextual categories. *Ann. Pure Appl. Logic*, 32(3):209–243, 1986.

[Fre25] Jonas Frey. Duality for clans: an extension of gabriel–ulmer duality. *J. Symb. Log.*, pages 1–38, 2025.

[GU71] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*, volume Vol. 221 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1971.

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium. Vol. 2*, volume 44 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, Oxford, 2002.

[Joy17] André Joyal. Notes on clans and tribes. *arXiv preprint arXiv:1710.10238*, 2017.

[Kaw24] Yuto Kawase. Relativized universal algebra via partial horn logic, 2024.

[MS82] John L. MacDonald and Arthur Stone. The tower and regular decomposition. *Cahiers Topologie Géom. Différentielle*, 23(2):197–213, 1982.

[MT82] John MacDonald and Walter Tholen. Decomposition of morphisms into infinitely many factors. In *Category theory (Gummersbach, 1981)*, volume 962 of *Lecture Notes in Math.*, pages 175–189. Springer, Berlin-New York, 1982.

[PV07] E. Palmgren and S. J. Vickers. Partial Horn logic and cartesian categories. *Ann. Pure Appl. Logic*, 145(3):314–353, 2007.

Thank you!

Hayato Nasu

Our preprint will be out soon.

Proof of $\delta(\mathrm{Mod}(\mathcal{E})) \leq \delta(\mathrm{Mod}(\mathcal{E}')) + 1$

A1 A3

Lemma. $\mathcal{K}, \mathcal{L}$: have finite limits & colimits , $\mathcal{K} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{L}$ ($F \dashv G$)

$f : X \to Y$ in $\mathcal{L}$ s.t. Gf : str. epi. , $\delta_{\mathcal{K}}(Gf)$ exists.

$$\begin{tikzcd} FGX \arrow[r, "FGf"] \arrow[d, "\varepsilon_X"'] & FGY \arrow[d] \arrow[ddr, bend left, "\varepsilon_Y"] & \\ X \arrow[r] \arrow[drr, bend right, "f"'] & E_f \arrow[ul, phantom, "\ulcorner"] \arrow[dr, "c_f"] & \\ & & Y \end{tikzcd}$$

something we already know

$$\delta_{\mathcal{L}}(f) \leq \delta_{\mathcal{K}}(Gf) + \delta_{\mathcal{L}}(c_f)$$

Example. $\mathrm{Set} \underset{\mathrm{Ob}}{\overset{(-)_\delta}{\rightleftarrows}} \mathrm{Cat}$, $\Phi : A \to B \rightsquigarrow$

$$\begin{tikzcd} (\mathrm{Ob}A)_\delta \arrow[r, "\mathrm{Ob}\Phi"] \arrow[d] & (\mathrm{Ob}B)_\delta \arrow[d] \arrow[ddr, bend left] & \\ A \arrow[r] \arrow[drr, bend right, "\Phi"'] & E(\Phi) \arrow[ul, phantom, "\ulcorner"] \arrow[dr] & \\ & & B \end{tikzcd}$$

Identification along with $\mathrm{Ob}\Phi$

The Ob-part is unchanged, reg. epi.

Why $\delta(c_f) \leq 1$?

A2 A3

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{L} \arrow[l, shift left, "G"] \end{tikzcd}$$

$$\begin{tikzcd} FGX \arrow[r, "FGf"] \arrow[d, "\varepsilon_X"'] & FGY \arrow[d, "r"'] \arrow[ddr, bend left, "\varepsilon_Y"] & \\ X \arrow[r] \arrow[drr, bend right, "f"'] & E_f \arrow[dr, "c_f"] & \\ & & Y \end{tikzcd}$$

- If G preserves pushouts and G : f.f. and then Gc_f is an iso, so c_f is in the image of $\pi_0 : G\downarrow^{\cong} GY \to \mathcal{L}$.
- This preserves reg. decompositions when G is a Street fibration, so $\delta(c_f) < \delta(G\downarrow^{\cong} GY)$

Example.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift left, "(-)_\delta"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} \arrow[l, shift left, "\mathrm{Ob}"] \end{tikzcd}$$

$\mathrm{Ob}\downarrow^{\cong} S \simeq \{\text{categories } \mathcal{C} \text{ s.t. } \mathrm{Ob}\,\mathcal{C} = S\}$

$\simeq \mathrm{Mod}(\mathbb{T}_S)$ $\leftarrow$ regular $(\delta \leq 2)$

$\mathbb{T}_S$

Sorts $\vdash \mathrm{Mor}(s,s')$ $(s,s' \in S)$

$\vdots$

No need for dependency

 A3

Simple reflective sub $\Rightarrow \delta(c_f) \leq 1$

Given a simple clan inclusion with a reflection $\begin{tikzcd} \mathcal{C}' \arrow[r, hook, shift right, "I"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "\Lambda"'] \end{tikzcd}$,

we have $\exists$

$$\begin{tikzcd}[column sep=huge] \mathrm{Mod}(\mathcal{C}') \arrow[r, bend left=60] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=60, "\Lambda^*"'] & \mathrm{Mod}(\mathcal{C}) \arrow[l, "I^*"'] \end{tikzcd},$$

and the adjunction fits into the situation in the previous slide.

This is a product clan!

For $M \in \mathrm{Mod}(\mathcal{C}')$, $I^* \downarrow^{\cong} M \simeq \mathrm{Mod}(\mathcal{C}[M])$ where

$$\begin{tikzcd} \int M \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & \int(M \circ \Lambda) \arrow[d] \\ 1 \arrow[r] & \mathcal{C}[M] \end{tikzcd}$$

bi-push-out

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{C}[M]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & \mathrm{Mod}(\int(M \circ \Lambda)) \arrow[d] & \simeq \mathrm{Mod}(\mathcal{C})/_{M \circ \Lambda} \arrow[d] \\ \mathrm{Mod}(1) \arrow[r] & \mathrm{Mod}(\int M) & \simeq \mathrm{Mod}(\mathcal{C}')/_M \end{tikzcd}$$

$\simeq 1$