

On the decomposition
of a strong epimorphism
into regular morphisms

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What is the most fundamental theorem
in (abstract) algebra?

A. The fundamental theorem of
.....
homomorphisms!

$$A/\ker(f) \cong \operatorname{Im}(f)$$

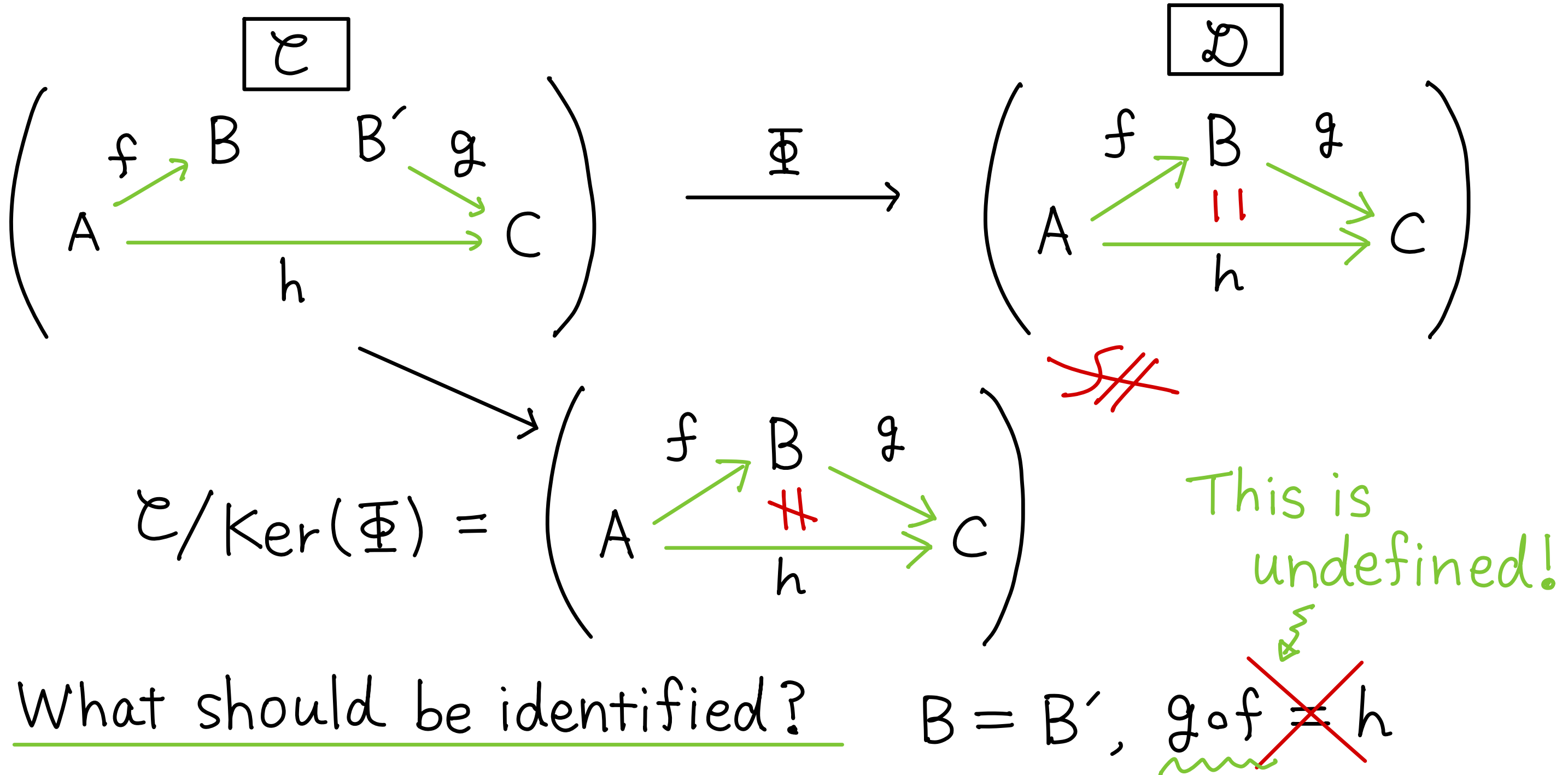
- For groups $\varphi: G \rightarrow H \rightsquigarrow G/\underline{\text{Ker}(\varphi)} \cong \text{Im}(\varphi)$
 $\uparrow$ normal subgroup
- For rings $\varphi: R \rightarrow S \rightsquigarrow R/\underline{\text{Ker}(\varphi)} \cong \text{Im}(\varphi)$
 $\uparrow$ ideal
- For modules $\varphi: M \rightarrow N \rightsquigarrow M/\underline{\text{Ker}(\varphi)} \cong \text{Im}(\varphi)$
 $\uparrow$ submodule

The common idea of $-/\text{Ker}(\varphi)$:

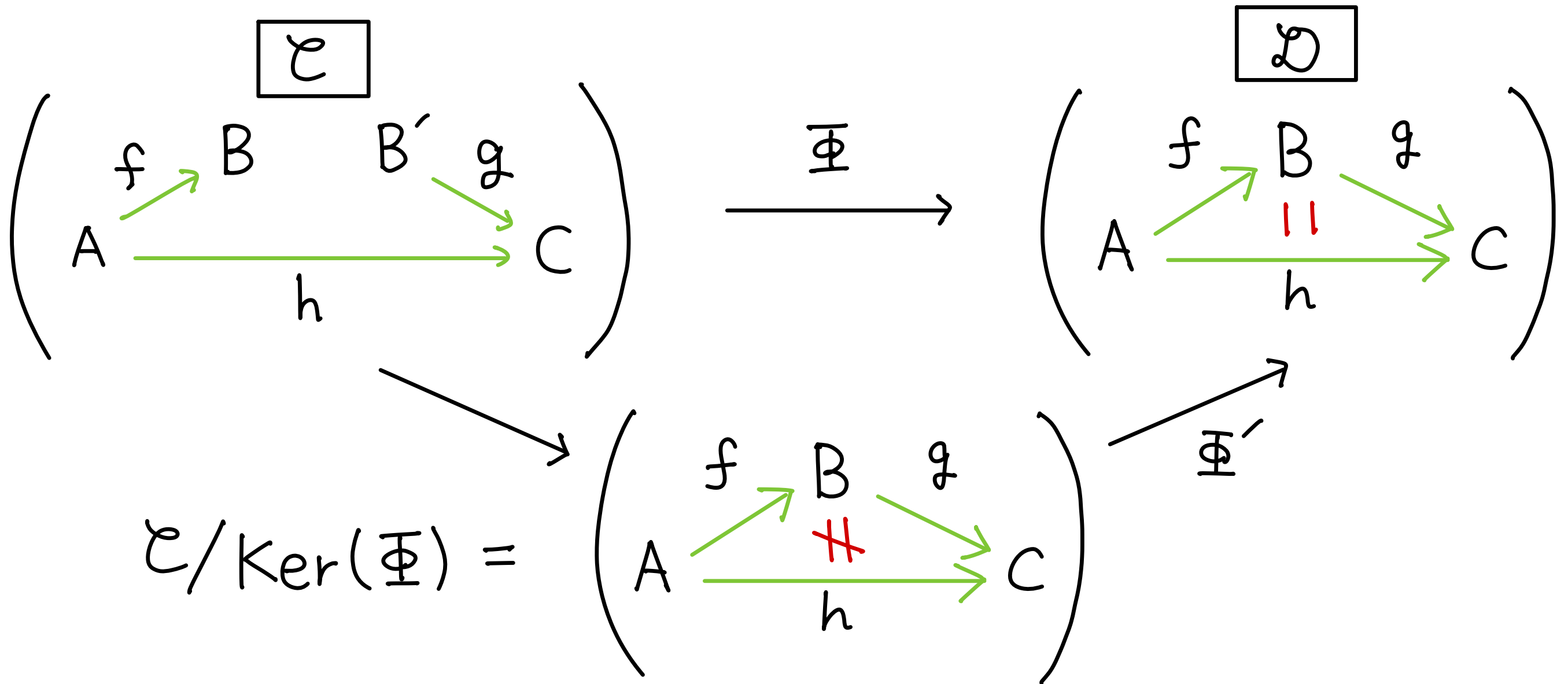
identifying two elements that go to the same element

What about categories? No!

Counter example :



Counter example :



$$\Rightarrow (\mathcal{C}/\text{Ker}(\Phi))/\text{Ker}(\Phi') \cong \text{Im}(\Phi)$$

1. The decomposition numbers
2. For models of generalized algebraic theories
3. For models of partial Horn theories

1. The decomposition numbers

2. For models of generalized algebraic theories

3. For models of partial Horn theories

Quotient by kernels

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A **regular epimorphism** is a coequalizer of some parallel pair.

Proposition. $\mathcal{C}$: a category w/ pullbacks

$f : A \rightarrow B$ is a regular epimorphism

$\Leftrightarrow \text{Ker}(f) \begin{smallmatrix} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{smallmatrix} A \xrightarrow{f} B$: coequalizer, where

$$\begin{array}{ccccc} \text{Ker}(f) & \xrightarrow{p_2} & A & & \\ p_1 \downarrow & \lrcorner & \downarrow f & & \\ A & \xrightarrow{f} & B & & \end{array}$$

Starting from $A \xrightarrow{f} B$, we obtain a sequence of reg. epis

$$\begin{array}{ccccccc} A & \twoheadrightarrow & A_1 & \twoheadrightarrow & A_2 & \twoheadrightarrow & A_3 \cdots \\ & & \searrow f_1 & & \searrow f_2 & & \searrow \cdots \\ & & & & & & B \end{array}$$

transfinitely

The image of a morphism

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$f : A \rightarrow B$ is a **strong epimorphism** if

$$\begin{array}{ccc} A & \xrightarrow{k} & X \\ f \downarrow & \text{red dashed } \exists! & \downarrow m \\ B & \xrightarrow{\ell} & Y \end{array} \quad \left(\begin{array}{l} \forall m : \text{mono} \\ \forall k \quad \forall \ell \end{array} \right)$$

When a category has
all pullbacks

$\longleftrightarrow A \xrightarrow{f} B$ factors through no proper $B' \rightarrow B$

If a morphism $f : A \rightarrow B$ can be presented as

$$A \xrightarrow[\text{str.epi.}]{k} I \xrightarrow[\text{mono}]{m} B,$$

then the factorization is unique up to isomorphism.

We call I **the image of f** .

reg.epi.

str.epi.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & \searrow & \uparrow \\
 A/\ker(f) & \xrightarrow[\exists!]{\cong} & \operatorname{Im}(f)
 \end{array}$$

When $\{\text{reg.epis}\} = \{\text{str.epis}\}$, this morphism is an iso.

Proposition. [Kelly '69, Gabriel, Ulmer '71]

In a locally presentable category, the transfinite sequence stabilizes at some λ , and gives

$$\begin{array}{ccccccc}
 A & \twoheadrightarrow & A_1 & \twoheadrightarrow & A_2 & \twoheadrightarrow & A_3 \dots \\
 & \searrow f & \searrow f_1 & \searrow f_2 & \searrow \dots & & \\
 & & & & & & B
 \end{array}$$

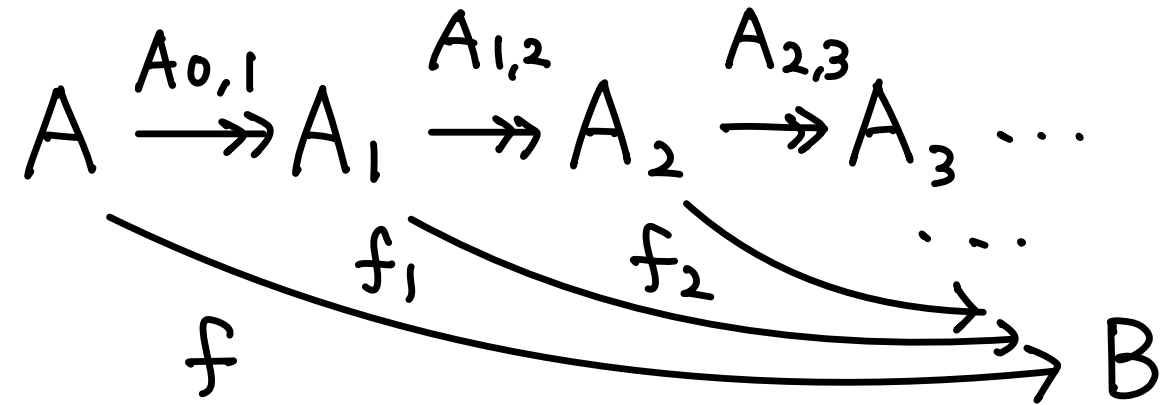
$$\begin{array}{ccc}
 A & \twoheadrightarrow & A_\lambda & \xrightarrow{\quad} & B \\
 \text{str.epi.} & \parallel & & \text{mono} & \\
 & \operatorname{Im}(f) & & &
 \end{array}$$

The decomposition number

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Definition. [Gabriel, Ulmer '71, B\"orger '91, KN.]

The **regular (canonical) decomposition number** $\delta(f)$ of $A \xrightarrow{f} B$ is the smallest α at which the seq. stabilizes ($A_{\alpha, \alpha+1} : \text{iso}$).



The **global regular (canonical) decomposition number** $\delta(\mathcal{C})$ of a category $\mathcal{C}$ is the smallest γ s.t. $\delta(f) \leq \gamma$ for any f in $\mathcal{C}$.

Proposition. [Gabriel, Ulmer '71, KN.]

$\mathcal{C} : \text{locally } \lambda\text{-presentable} \Rightarrow \delta(\mathcal{C}) \leq \lambda + 1.$

Examples

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• $\text{Grp}, \text{Ring}, \text{Module}_R : \delta = 2$

$$\text{Ker}(f) \rightrightarrows A \xrightarrow{f} B$$

$\downarrow \qquad \uparrow$

• Any regular category : $\delta \leq 2$

$$A/\text{Ker}(f) \cong \text{Im}(f)$$

• $\delta(\text{Pos}) = 2$

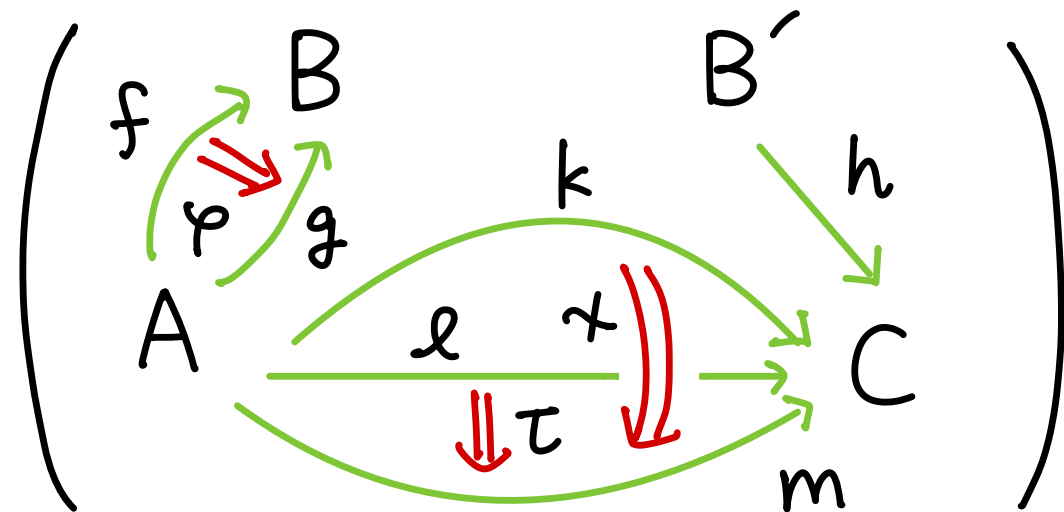
• $\delta(\text{Field}) = 1 \quad (\because K \cong \text{Im}(f))$

• [Bednarczyk, et.al. '99] $\delta(\text{Cat}) = 3$

Any strong epi is the composite of two regular epis.

$$\left(\Phi: A \rightarrow B \text{ is a str. epi} \Leftrightarrow \text{surj. on obj.} + \begin{array}{l} \forall g \text{ in } B \quad \exists f_1, \dots, f_n \\ g = \Phi(f_n) \circ \dots \circ \Phi(f_1) \end{array} \right)$$

$$\delta(n\text{-Cat}) = ?$$



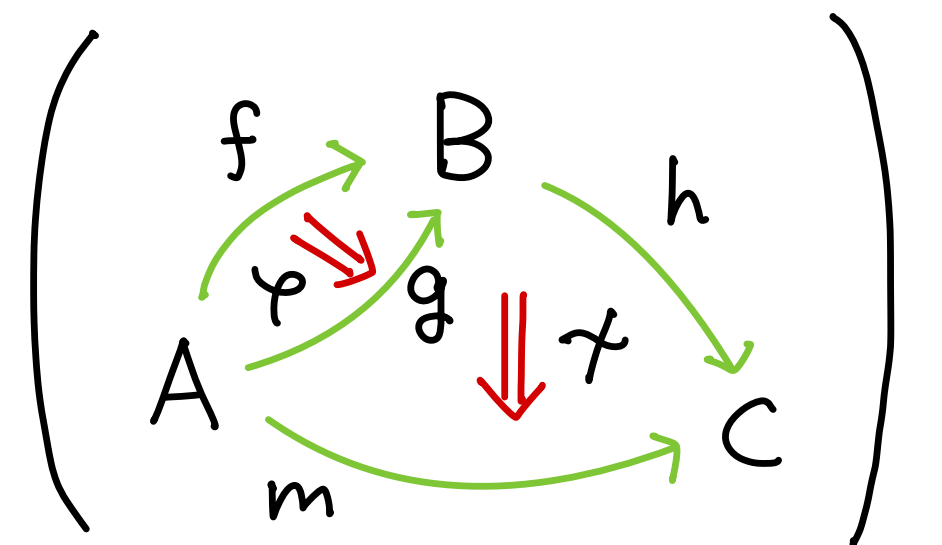
$$\xrightarrow{\Phi}$$

$$B' \mapsto B$$

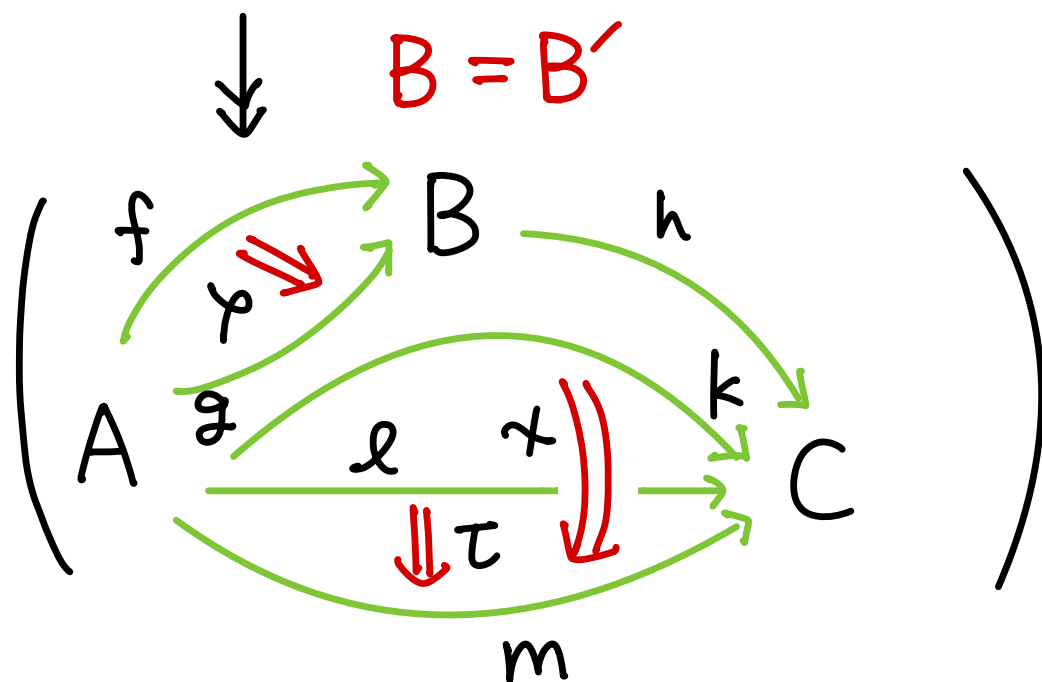
$$k \mapsto h \circ g$$

$$l \mapsto h \circ f$$

$$\tau \mapsto \gamma \circ (h\varphi)$$



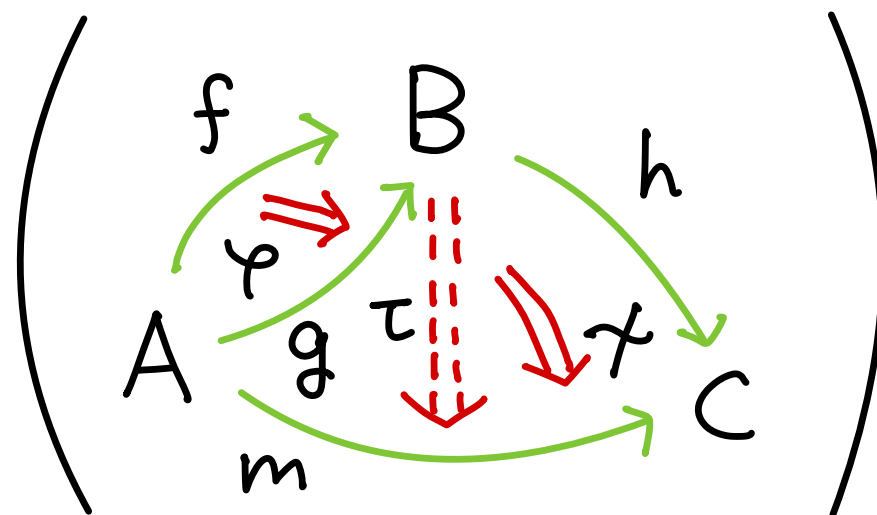
$$\Uparrow \tau = \gamma \circ (h\varphi)$$



$$\Rightarrow$$

$$h \circ f = l$$

$$h \circ g = k$$



$$\delta(\Phi) = 3 \rightsquigarrow \delta(2\text{-Cat}) \geq 4.$$

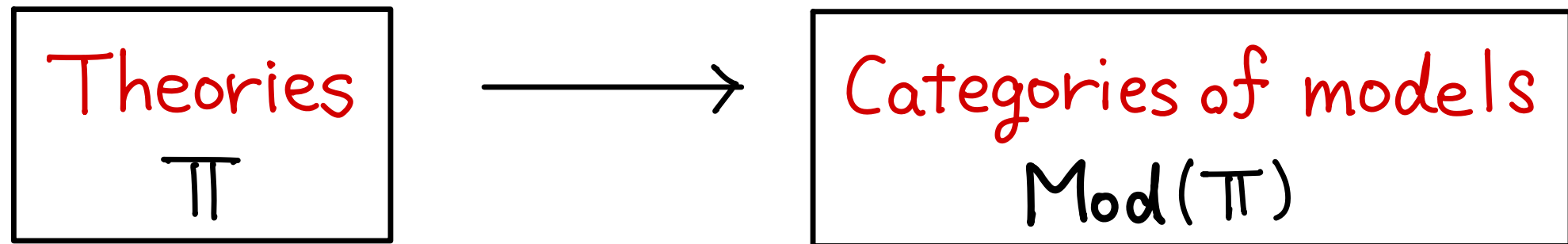
Is this = ?

1. The decomposition numbers
2. For models of generalized algebraic theories
3. For models of partial Horn theories

$$\text{Grp} \simeq \left\{ \left(X, \begin{array}{c} x^2 \dot{\rightarrow} x, \\ 1 \xrightarrow{e} x, x \xrightarrow{(-)^{-1}} x \end{array} \right) \middle| \begin{array}{c} \text{group} \\ \text{axioms} \end{array} \right\} \simeq \text{FPCAT}(\underbrace{\pi_{\text{Grp}}}_{\substack{\uparrow \\ \text{The Lawvere theory} \\ \text{of groups}}}, \text{Set})$$

$\text{Mod}(\pi_{\text{Grp}})$!!

Proposition. $\pi : \text{Lawvere theory} \Rightarrow \text{Mod}(\pi) : \text{regular}$



satisfies $P_n \implies \delta \leq n$

Equational theory $\approx$ Lawvere theory $\xrightarrow{\text{Mod}}$ Categories of algebras
 (operations + equational axioms) $\parallel$ $\text{FPCAT}(-, \text{Set})$

Cartesian theory [Johnstone '02] $\approx$ Categories with finite limits $\xrightarrow{\text{Mod}}$

Essentially algebraic theory [Freyd '72] $\xrightarrow{\text{Mod}}$ Locally finitely presentable categories
 $\xrightarrow{\text{Mod}}$

partial Horn logic [Palmgren, Vickers '07]

generalized algebraic theory [Cartmell '78] $\approx$ clans [Joyal '17] $\xrightarrow{\text{Clan}(-, \text{Set})}$

Generalized algebraic theory

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(1-sorted) algebras

$\Downarrow$

many-sorted algebras

e.g. Module : the cat. of pairs (R, M)
 $\parallel$
 $\text{Mod}(\Pi_{\text{Module}})$
ring $\nearrow$ R -module $\uparrow$

Π_{Module}

Sorts $\vdash R, \vdash M$

Operators $a:R, b:R \vdash a \circ b : R$
 $a:R, x:M \vdash a \circ x : M$
 $\vdots$

Axioms $a:R, b:R, x:M$
 $\vdash (a \circ b) \circ x = a \circ (b \circ x) : M$
 $\vdots$

Generalized algebraic theory

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1-sorted algebras

$\Downarrow$

many-sorted algebras

$\Downarrow$

dependent-sorted algebras

e.g. $\text{Cat} = \text{Mod}(\Pi_{\text{cat}})$

Π_{cat}

Sorts

$\vdash \text{Ob}$

$x : \text{Ob}, y : \text{Ob} \vdash \text{Mor}(x, y)$

Operators

$x : \text{Ob} \vdash \text{id}(x) : \text{Mor}(x, x)$

$x, y, z : \text{Ob}, f : \text{Mor}(x, y), g : \text{Mor}(y, z)$

$\vdash \text{comp}(x, y, z, f, g) : \text{Mor}(x, z)$

$\hookrightarrow g \circ f$

Axioms

$x : \text{Ob}, f : \text{Mor}(x, y) \vdash f \circ \text{id}(x) = f$

$: \text{Mor}(x, y)$

Π_{Module}

Sorts $\vdash R, \vdash M$

Operators $a : R, b : R \vdash a \circ b : R$

$a : R, x : M \vdash a \circ x : M$

$\vdots$

Axioms $a : R, b : R, x : M$

$\vdash (a \circ b) \circ x = a \circ (b \circ x) : M$

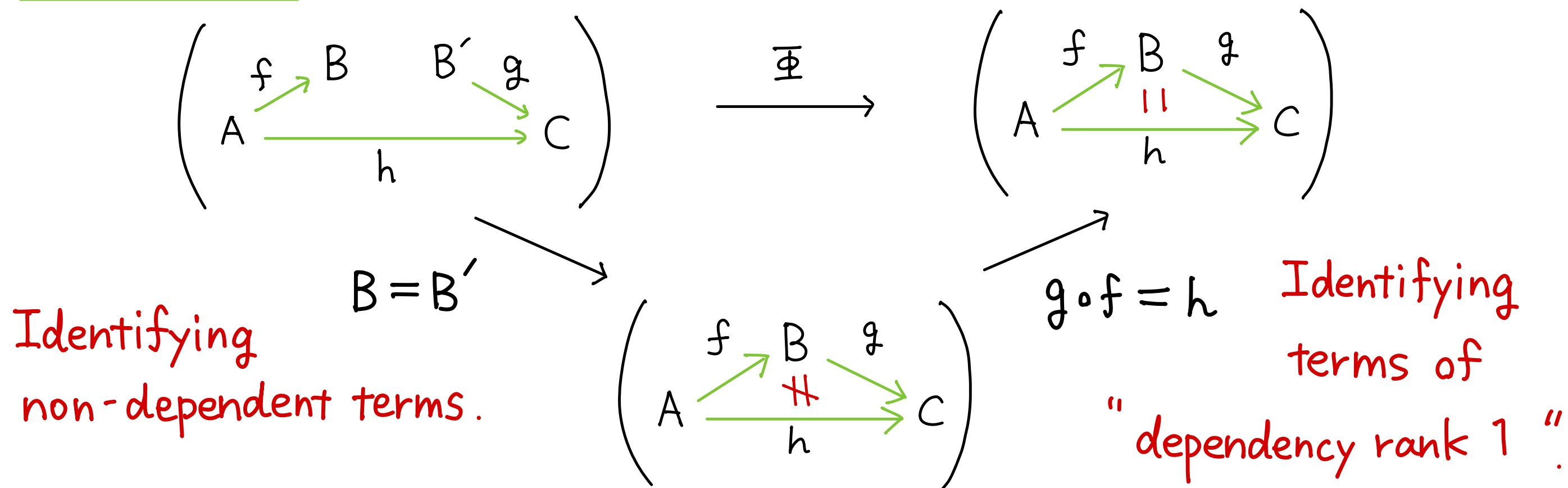
$\vdots$

Proposition. [Cartmel '78, Adámek, Rosický '94]

$\mathcal{C}$: locally finitely presentable $\Leftrightarrow \exists \pi : \text{GAT} \quad \mathcal{C} \simeq \text{Mod}(\pi)$

Given a GAT π , how can we know $\delta(\text{Mod}(\pi))$?

Observation.



The dependency rank $\rightsquigarrow \delta$

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Definition. In a GAT Π , we define **the dependency rank** $dr(A)$

for each well-formed type $\Gamma \vdash A$ by

$$dr(A) = \max \{ dr(B) \mid B \text{ appear in } \Gamma \} + 1.$$

Example. $dr(Ob) = 1$, $dr(Mor(x, y)) = 2$

We call Π a **non-descending** if

- $\Gamma \vdash f(\vec{x}) : B$: operator symbol
- $\Gamma \vdash t = s : B$: axiom

$$\Rightarrow dr(B) \geq \max_{A \text{ in } \Gamma} dr(A)$$

Theorem. [KN.] Π : non-descending, $\max_{A \text{ in } \Pi} dr(A) \leq n$

$\Rightarrow \forall g : M \rightarrow N$ in $\text{Mod}(\Pi)$ $\delta(f) \leq n+1$, hence $\delta(\text{Mod}(\Pi)) \leq n+2$

Examples

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- $n\text{Cat} = \text{Mod}(\Pi_{n\text{cat}})$

Sorts $\vdash \underline{0}, x, x' : \underline{0} \vdash \underline{1}(x, x'), x, x' : \underline{0}, y, y' : \underline{1}(x, x') \vdash \underline{2}(x, x', y, y')$

Operators $x, x' : \underline{0}, y, y', y'' : \underline{1}(x, x'), z : \underline{2}(x, x', y, y'), z' : \underline{2}(x, x', y', y'') \dots$
 $\vdash v\text{-comp}(x, x', y, y', y'', z, z') : \underline{2}(x, x', y, y'') \dots$

$\text{dr}(\underline{n}) = n + 1$, so $\delta(n\text{Cat}) \leq n + 2$. In fact, $\delta(n\text{Cat}) = n + 2$.

- $\delta(\text{strMonCat}) = 3$.

- $\delta(\text{strDblCat}) = 4$.

← not non-descending!

- $\exists \Pi : \text{GAT}$ with $\max\{\text{dr}(A)\} = 1$. $\delta(\text{Mod}(\Pi)) = \omega + 1$.

$\max_{A \text{ in } \Pi} \text{dr}(A) = n$, Π : non-descending

$\Rightarrow \Pi_0 \longrightarrow \Pi_1 \longrightarrow \cdots \longrightarrow \Pi_n = \Pi$: a seq. of theories
with $\delta(\text{Mod}(\Pi_{i+1})) \leq \delta(\text{Mod}(\Pi_i)) + 1 \quad (\forall i)$

Example. $\Pi. \longrightarrow \Pi_{\text{set}} \longrightarrow \Pi_{\text{cat}} \longrightarrow \Pi_{2\text{cat}} \longrightarrow \cdots \longrightarrow \Pi_{n\text{cat}}$

We derive a sequence of their syntactic categories

$$\mathcal{C}_{\Pi_0} \hookrightarrow \mathcal{C}_{\Pi_1} \hookrightarrow \cdots \hookrightarrow \mathcal{C}_{\Pi_n} = \mathcal{C}_{\Pi}.$$

so the problem reduces to find a nice property of these functors.

The syntactic categories are seen as **clans**.

Definition. A **clan** is a pair of

- s.t. {
- a category $\mathcal{C}$, and
 - a class of morphisms D
 - D : closed under comp. contains all isos.
 - $\mathcal{C}$ has a terminal.
 - $! : C \rightarrow I \in D \quad (\forall C)$.
 - $\mathcal{C}$ has p.b.s along maps in D .
 - $D \ni \begin{array}{ccc} P & \xrightarrow{\quad} & C_3 \\ \downarrow \lrcorner & & \downarrow \\ C_1 & \longrightarrow & C_2 \end{array} \in D$.

Example. $\Pi : \text{GAT} \leadsto (\mathcal{C}_\Pi, D_\Pi)$

objects : contexts

morphisms : "tuples of terms"

$D_\Pi := \{ \text{projections } \Gamma, \Delta \rightarrow \Gamma \}$

$\text{Mod}(\Pi) = \text{Clan}((\mathcal{C}_\Pi, D_\Pi), \text{Set})$

Definition. $\text{Mod}(\mathcal{C}_\Pi)$

A **clan morphism** $\Phi : (\mathcal{C}, D) \rightarrow (\mathcal{C}', D')$

is a functor $\Phi : \mathcal{C} \rightarrow \mathcal{C}'$ s.t.

• $\Phi(D) \subseteq D'$

• Φ preserves the limits in Def.

We call a clan morphism $\Phi : (\mathcal{C}, D) \rightarrow (\mathcal{C}', D')$ **simple** if D' is the "smallest clan str." on $\mathcal{C}'$ that includes

$$\{d : C' \rightarrow \Phi(C) \text{ in } \mathcal{C}' \mid d \in D\}.$$

Example. • $\mathcal{C}_{\text{set}} \hookrightarrow \mathcal{C}_{\text{cat}}$ is simple.

• $\mathcal{C}_{\text{set}} \hookrightarrow \mathcal{C}_{2\text{cat}}$ is not simple.

removing types
with $dr > i$.

For the sequence $\pi_0 \rightarrow \dots \rightarrow \pi_n$, $\mathcal{C}_{\pi_i} \xrightarrow[\perp]{\exists} \mathcal{C}_{\pi_{i+1}}$ is simple.

Proposition. $\mathcal{C}' \xleftarrow[\perp]{F} \mathcal{C}$: vari in the 2-category of clans.

$$G : \text{simple} \Rightarrow \delta(\text{Mod}(\mathcal{C})) \leq \delta(\text{Mod}(\mathcal{C}')) + 1$$

1. The decomposition numbers
2. For models of generalized algebraic theories
3. For models of partial Horn theories

partial Horn logic

Operators are allowed to be partial.

Axioms are allowed to be Horn clauses, not just equations.

e.g. $xy = xz \vdash y = z$

Example (categories)

$\left(\begin{array}{l} \text{Ob}, \text{Mor}, s: \text{Mor} \rightarrow \text{Ob}, t: \text{Mor} \rightarrow \text{Ob}, \\ \text{id}: \text{Ob} \rightarrow \text{Mor}, - \circ -: \text{Mor} \times \text{Mor} \rightarrow \text{Mor} \end{array} \right) \text{ s.t. } \vdash s(x) \downarrow, \text{ is defined}$
 $\vdash \text{id}(t(x)) \circ x = x,$
 $\vdots$

$\leadsto \Pi_{\text{cat}} : \text{partial Horn theory of categories},$

$\text{Cat} \cong \text{Mod}(\Pi_{\text{cat}})$

Proposition. [Palmgren, Vickers '07, Kawase '23]

$\mathcal{C} : \text{locally finitely presentable} \iff \exists \Pi : \text{partial Horn th. } \mathcal{C} \cong \text{Mod}(\Pi)$

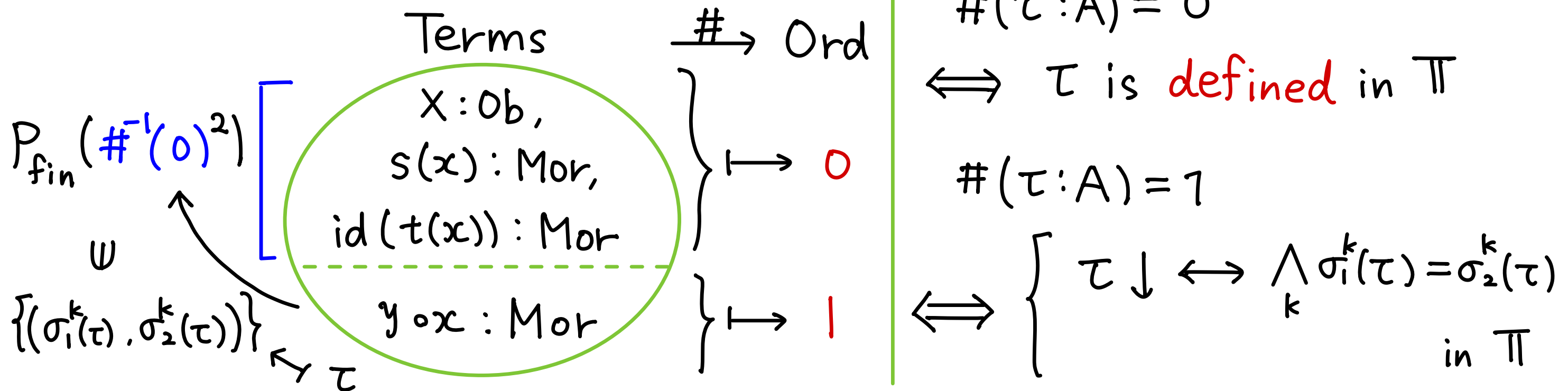
Upper bound of δ using PHL (Sketch)

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Theorem [KN.] If Π comes with "a gauge of length γ ",
 for any $f: M \rightarrow N$ in $\text{Mod}(\Pi)$, $\delta(f) \leq \gamma$. Thus, $\delta(\text{Mod}(\Pi)) \leq \gamma + 1$.

Idea: Layering all terms based on definedness.

Example (a gauge $(\#, \sigma)$ for Π_{cat})



GAT approach vs PHL approach

- GAT gives a simpler way to determine the upper bound of δ for a limited case.

- PHL approach is more general (even for locally λ -presentable categories), but finding a gauge is not a simple task.

Omitted topics ([KN.])

- The regular decomposition gives the most efficient way to present a str.epi as a composite of reg.epis

$$\begin{array}{c}
 \xrightarrow{\quad f \quad} \\
 \parallel \\
 \xrightarrow{f_1} \xrightarrow{f_2} \dots \xrightarrow{f_n} \\
 \text{reg. epis}
 \end{array}
 \Rightarrow \delta(f) \leq n$$

- Generalization of the class of regular epimorphisms (localization functors in $\mathbf{Cat}$, etc.)

Summary

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- The fundamental theorem of homomorphisms states that **strong epis $\Leftrightarrow$ regular epis** in the category in question.
- They do not coincide in general, but in l.p. categories.
strong epis $\Leftrightarrow$ transfinite composites of regular epis.
- The number of regular epis needed (**= decomposition number**) can have an upper bound determined by GAT/PHL presentation.
 - GAT** The dependency rank + 2 (when non-descending)
 - PHL** The length of a gauge + 1

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Thank you!

Hayato Nasu

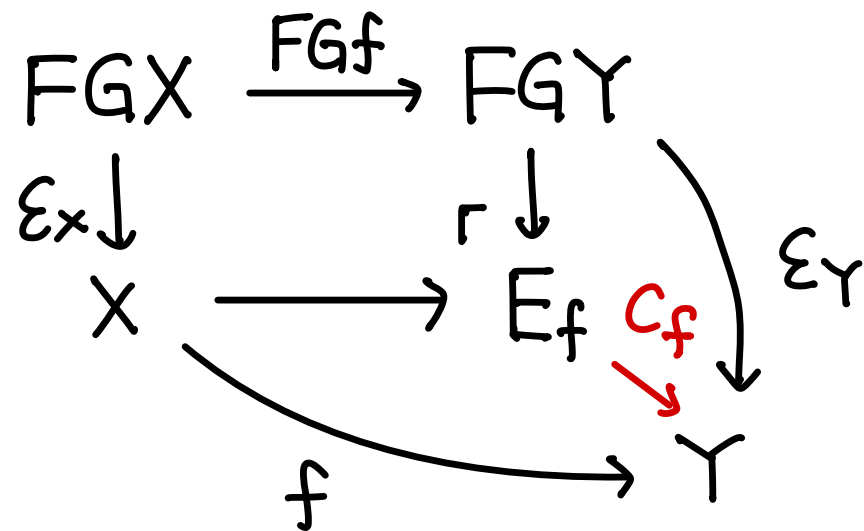
Our preprint will be out soon.

Proof of $\delta(\text{Mod}(\mathcal{C})) \leq \delta(\text{Mod}(\mathcal{C}')) + 1$

A1 A3

Lemma. $\mathcal{K}, \mathcal{L}$: have finite limits & colimits , $\mathcal{K} \xrightleftharpoons[G]{F} \mathcal{L}$

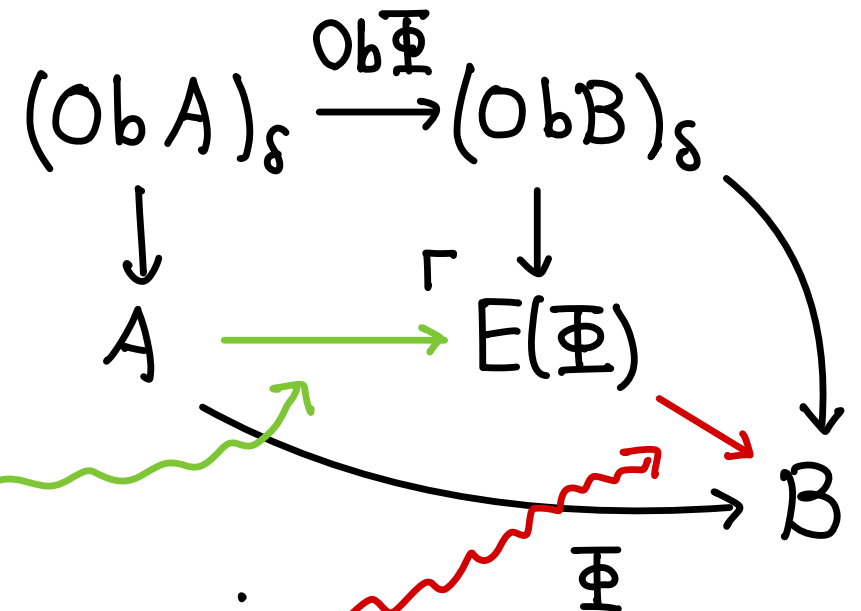
$f: X \rightarrow Y$ in $\mathcal{L}$ s.t. Gf : str. epi. , $\delta_{\mathcal{K}}(Gf)$ exists.



something we already know

$$\delta_{\mathcal{L}}(f) \leq \delta_{\mathcal{K}}(Gf) + \delta_{\mathcal{L}}(c_f)$$

Example. $\text{Set} \xrightleftharpoons[\text{Ob}]{(-)_\delta} \text{Cat}$, $\Phi: A \rightarrow B \rightsquigarrow$



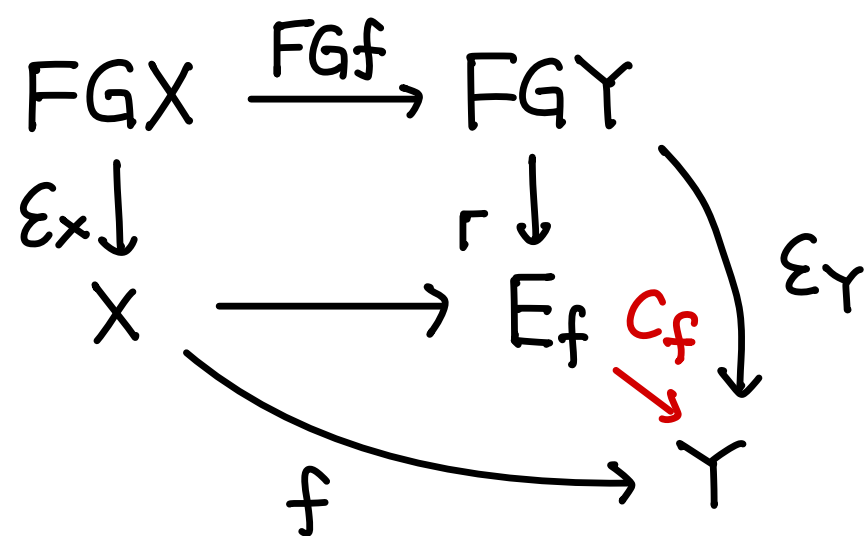
Identification along with $\text{Ob} \Phi$

The Ob -part is unchanged, reg. epi.

Why $\delta(c_f) \leq 1$?

A2 A3

$$\mathcal{K} \xrightleftharpoons[G]{F} \mathcal{L}$$



• If G preserves pushouts and $G: \mathcal{K} \rightarrow \mathcal{L}$ is a fibration, then Gc_f is an iso, so c_f is in the image of $\pi_0: G \downarrow \cong GY \rightarrow \mathcal{L}$.

• This preserves reg. decompositions when G is a Street fibration, so $\delta(c_f) < \delta(G \downarrow \cong GY)$

Example. $\text{Set} \xrightleftharpoons[\text{Ob}]{(-)_\delta} \text{Cat}$

$$\begin{aligned} \text{Ob} \downarrow \cong S &\simeq \{ \text{categories } \mathcal{C} \text{ s.t. } \text{Ob } \mathcal{C} = S \} \\ &\simeq \text{Mod}(\Pi_S) \quad \text{regular } (\delta \leq 2) \end{aligned}$$

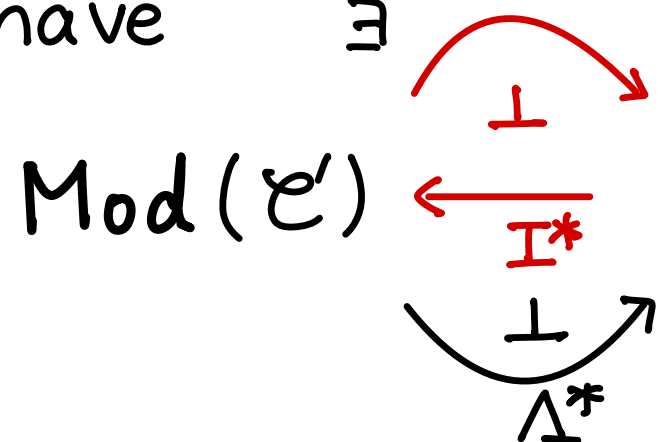
$$\begin{array}{c} \Pi_S \\ \text{Sorts} \vdash \text{Mor}(s, s') \\ \quad \quad \quad (s, s' \in S) \\ \uparrow \quad \quad \vdots \end{array}$$

No need for dependency

Simple reflective sub $\Rightarrow \delta(c_f) \leq 1$

A3 A3

Given a simple clan inclusion with a reflection $\mathcal{C}' \xrightleftharpoons[I]{\Lambda} \mathcal{C}$,
we have $\exists$



and the **adjunction** fits into the situation in the previous slide.

For $M \in \text{Mod}(\mathcal{C}')$, $I^* \downarrow \cong M \simeq \text{Mod}(\mathcal{C}[M])$ where
 This is a product clan!

$$\begin{array}{ccc} \int M & \longrightarrow & \int (M \circ \Lambda) \\ \downarrow & \nearrow \cong & \downarrow \\ 1 & \longrightarrow & \mathcal{C}[M] \end{array}$$

bi-push-out

$$\begin{array}{ccccc} \text{Mod}(\mathcal{C}[M]) & \longrightarrow & \text{Mod}(\int (M \circ \Lambda)) & \simeq & \text{Mod}(\mathcal{C}) / M \circ \Lambda \\ \downarrow & \lrcorner \cong & \downarrow & & \downarrow \\ \text{Mod}(1) & \longrightarrow & \text{Mod}(\int M) & \simeq & \text{Mod}(\mathcal{C}') / M \\ & \simeq 1 & & & \end{array}$$