(Hyper)doctrines as Virtual Double Categories

Hayato Nasu
RIMS, Kyoto

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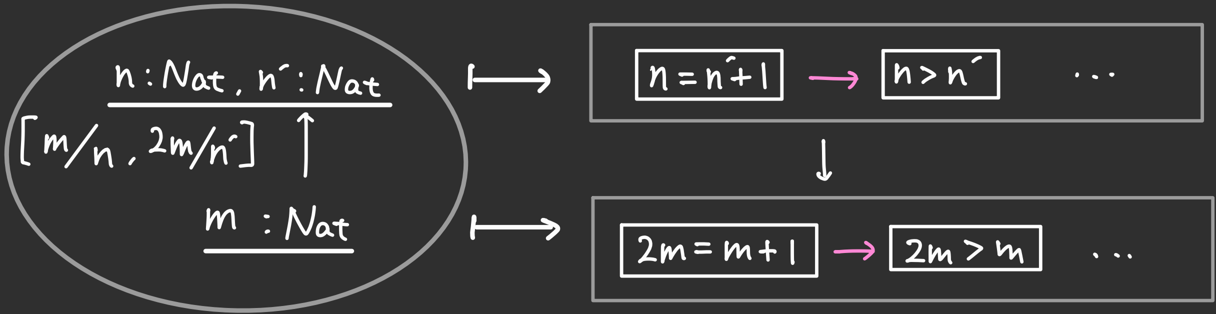
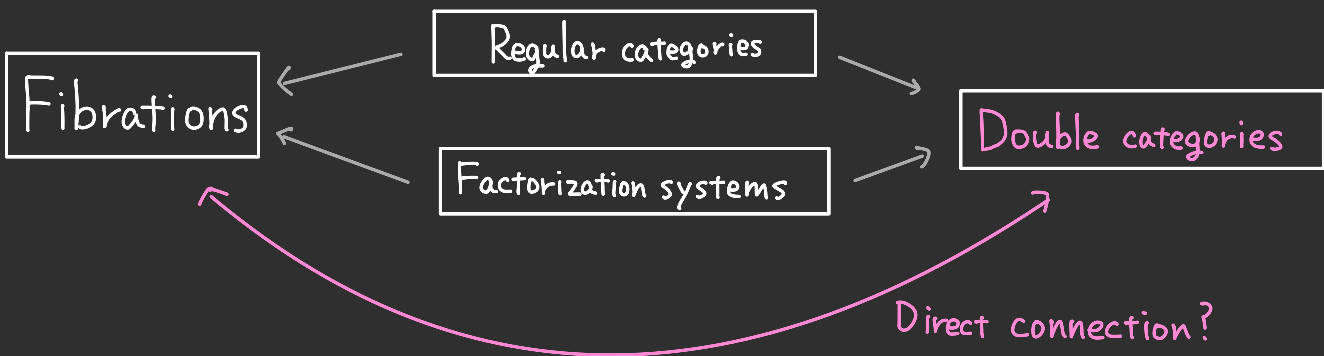
Introduction

A (hyper)doctrine $=$ a pseudo-functor $P : \mathcal{C}^{op} \longrightarrow \mathbf{Preorder}$.



A fibration $\fallingdotseq$ a pseudo-functor $P : \mathcal{C}^{op} \longrightarrow \mathbf{CAT}$

$=$ a proof relevant doctrine



Structures

1. Double categories of relations relative to factorisation systems
   - 1.1. Double categories : definition and examples
   - 1.2. A characterization theorem
2. Virtual double categories and fibrations.
   - 2.1. Making fibrant VDCs from primary fibrations.
   - 2.2. $\exists$ in virtual double categories.

The 1st half is based on joint work with Keisuke Hoshino available on ArXiv:

Hoshino, N., "Double categories of relations relative to factorisation systems" (2023)

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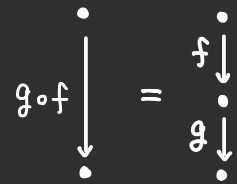
Double categories

A **double category** $\mathbb{D}$ consists of the following data:

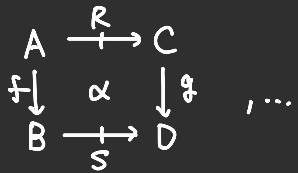
- **objects** $A, B, C, \ldots$
- **vertical arrows** $f\colon A \downarrow B, \ldots$ and their composition



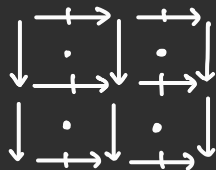
- **horizontal arrows** $A \overset{R}{\nrightarrow} B, \ldots$ and their composition

$$\bullet \xrightarrow{R\odot S} \bullet := \bullet \xrightarrow{R} \bullet \xrightarrow{S} \bullet$$

- **cells**

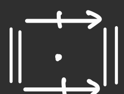


with appropriate data of composition and axioms.



In particular,

- (objects, vertical arrows) gives a category (the vertical category), and
- (objects, horizontal arrows, horizontal cells) gives a bicategory (the horizontal bicategory)



Examples of double categories

Rel : the double category of sets, functions, and relations

composition of horizontal arrows :

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$$

$$:= \left\{ (a,c) \;\middle|\; \exists b{:}B.\ \begin{matrix} (a,b) \in R \\ (b,c) \in S \end{matrix} \right\}$$

identity horizontal arrows :

$$A \nrightarrow A$$

$$:= \{ (a, a') \mid a = a' \}$$

cells :

$$\begin{array}{ccc} A & \overset{R}{\nrightarrow} & C \\ f \downarrow & \wedge\!| & \downarrow g \\ B & \underset{S}{\nrightarrow} & D \end{array}$$

$$\iff \quad (a,c) \in R \Rightarrow (fa, gc) \in S$$

Span : the double category of sets, functions, and spans

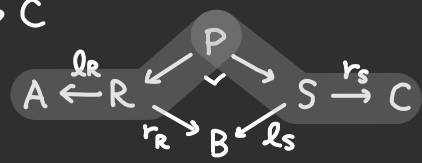
A span from A to B consists of

$$A \overset{l_R}{\longleftarrow} R \overset{r_R}{\longrightarrow} B$$

composition of horizontal arrows :

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$$

$:=$



cells :

$$\begin{array}{ccc} A & \overset{R}{\nrightarrow} & C \\ f \downarrow & \alpha & \downarrow g \\ B & \underset{S}{\nrightarrow} & D \end{array} \quad \Bigg\| \quad \begin{array}{ccccc} A & \overset{l_R}{\longleftarrow} & R & \overset{r_R}{\longrightarrow} & C \\ f \downarrow & \circlearrowleft & \downarrow \alpha & \circlearrowleft & \downarrow g \\ B & \underset{l_S}{\longleftarrow} & S & \underset{r_S}{\longrightarrow} & D \end{array}$$

Features of double categories of relations / spans

- A restriction $R(f;g)$ of

$$\begin{array}{ccc} A & \xrightarrow{R(f;g)} & C \\ f\downarrow & \text{cart.} & \downarrow g \\ B & \xrightarrow[R]{} & D \end{array}$$

is the universal cell of this form:

$$\begin{array}{ccc} X & \xrightarrow{Q} & Y \\ h\downarrow & & \downarrow k \\ A & \alpha & C \\ f\downarrow & & \downarrow g \\ B & \xrightarrow[R]{} & D \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{Q} & Y \\ h\downarrow & \exists!\ \tilde{\alpha} & \downarrow k \\ A & \xrightarrow{R(f;g)} & C \\ f\downarrow & \text{cart.} & \downarrow g \\ B & \xrightarrow[R]{} & D \end{array}.$$

In $\mathbb{R}\mathrm{el}$,

$$\begin{array}{ccc} A & \xrightarrow{R(f;g)} & C \\ f\downarrow & \wedge\!| & \downarrow g \\ B & \xrightarrow[R]{} & D \end{array} := \{(a,c) \mid (fa, gc) \in R\}$$

- A double category $\mathbb{D}$ is cartesian ($=$ has finite products) if $\mathbb{D} \xrightarrow{\Delta} \mathbb{D}^n$ has a right adjoint in $\mathbb{D}\mathrm{bl}$. $\mathbb{R}\mathrm{el}$ is cartesian with the obvious cartesian structure.

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Double categories of relations relativized

Relations are a special kind of spans: jointly monic spans.

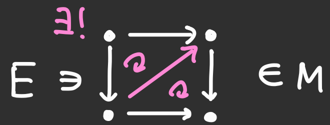
$$A \xleftarrow{\ell} R \xrightarrow{r} B \text{ is a relation} \iff R \xrightarrow{\langle \ell, r\rangle} A \times B \in \underline{\text{Mono}}$$

Replacing <u>Mono</u> with other classes of morphisms, we have a variety of notions of relations.

An **orthogonal factorization system (OFS)** on a category $\mathcal{C}$ is a pair of classes of morphisms (E, M) such that:

(i) E and M are closed under composition and contain iso's.

(ii) E and M are orthogonal:



(iii) Every morphism in $\mathcal{C}$ is factored as $\cdot \xrightarrow{\in E} \cdot \xrightarrow{\in M} \cdot$.

It is called **stable** if E is stable under pullback.

Double categories of relations relativized

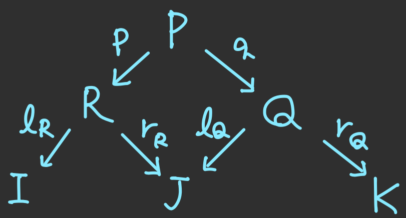
From an SOFS (E, M) on a category $\mathcal{C}$, we construct the double category $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ consisting of the objects, the arrows, and the M-relations.

$$\begin{array}{ccc} I & \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & K \\ f\downarrow & \alpha & \downarrow g \\ J & \underset{S}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & L \end{array} \quad \Bigg\| \quad \begin{array}{ccccc} & \overset{l_R}{\swarrow} & R & \overset{r_R}{\searrow} & \\ I & \circlearrowleft & \downarrow\alpha & \circlearrowright & K \\ f\downarrow & & & & \downarrow g \\ J & \underset{l_S}{\swarrow} & S & \underset{r_S}{\searrow} & L \end{array} \qquad \text{with} \begin{cases} \langle l_R, r_R\rangle : R \longrightarrow I\times K \\ \langle l_S, r_S\rangle : S \longrightarrow J\times L \end{cases} \in M$$

The composite of $I \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} J \overset{Q}{\longrightarrow\!\!\!\!\!\!\!+\;\;} K$ is defined in the following way.

(1) Take the pullback

$$\begin{array}{ccccc} & \overset{p}{\swarrow} & P & \overset{q}{\searrow} & \\ R & & \lrcorner & & Q \\ & \underset{r_R}{\searrow} & J & \underset{l_Q}{\swarrow} & \end{array} .$$



(2) Factorize as

$$\begin{array}{ccccc} P & \xrightarrow{\langle p, q\rangle} & R\times Q & \xrightarrow{l_R \times r_Q} & I\times K \\ & {}_{E \ni}\searrow & P' & \nearrow_{\langle l', r'\rangle \in M} & \end{array} ,$$

and define $I \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} J \overset{Q}{\longrightarrow\!\!\!\!\!\!\!+\;\;} K := I \overset{P'}{\longrightarrow\!\!\!\!\!\!\!+\;\;} K$.

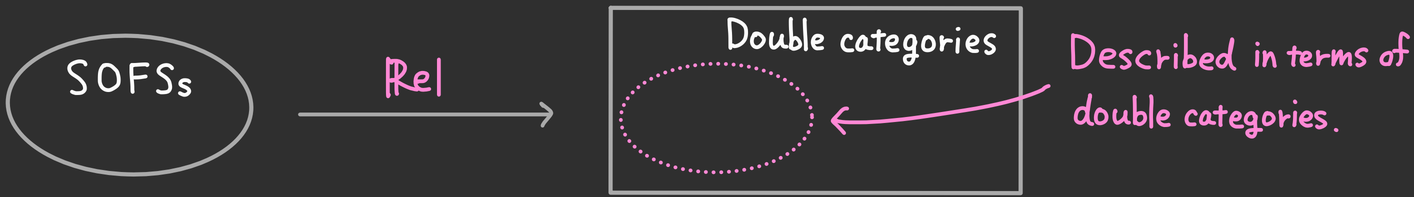
Double categories of relations relativized

Theorem [HN23] For a double category $\mathbb{D}$, the following are equivalent:

(1) $\mathbb{D} \simeq \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ for some SOFS (E,M) on some finitely complete category $\mathcal{C}$.

(2) $\mathbb{D}$ is a cartesian equipment with Beck-Chevalley pullbacks and admits the M-comprehension scheme.

If these hold, the classes M in these are the same.



$\mathcal{C}$	Regular categories	Lex categories	Quasitoposes
(E, M)	(RegEpi, Mono)	(Iso, All)	(Epi, StrongMono)
$\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$	DC of relations	DC of spans	DC of strong relations

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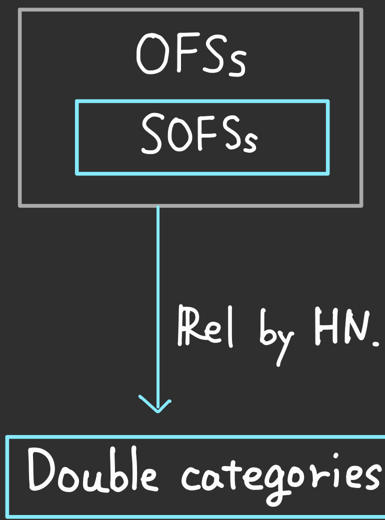
Some frameworks and their connection



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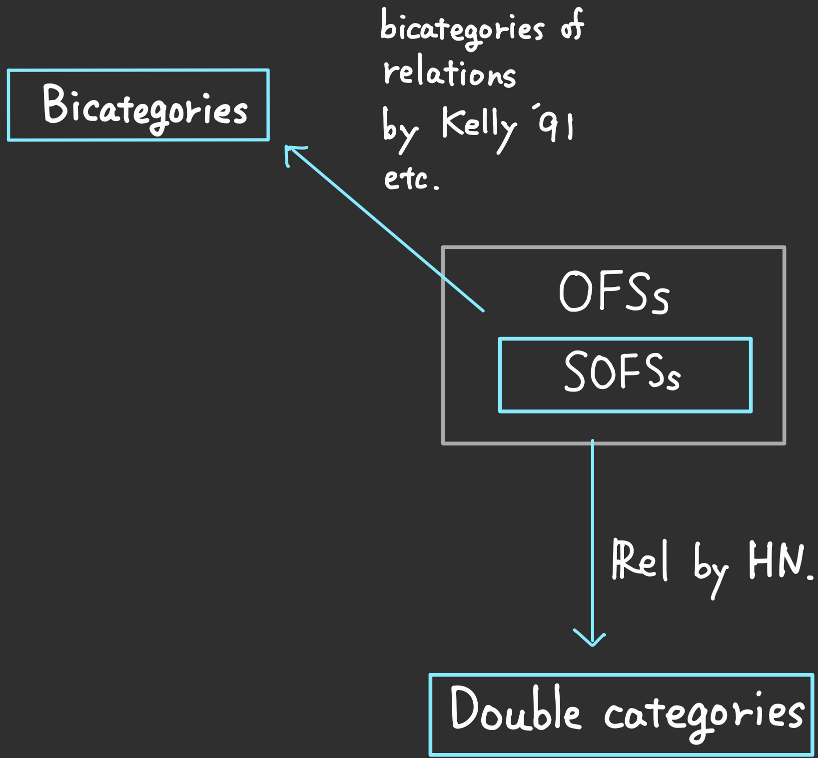

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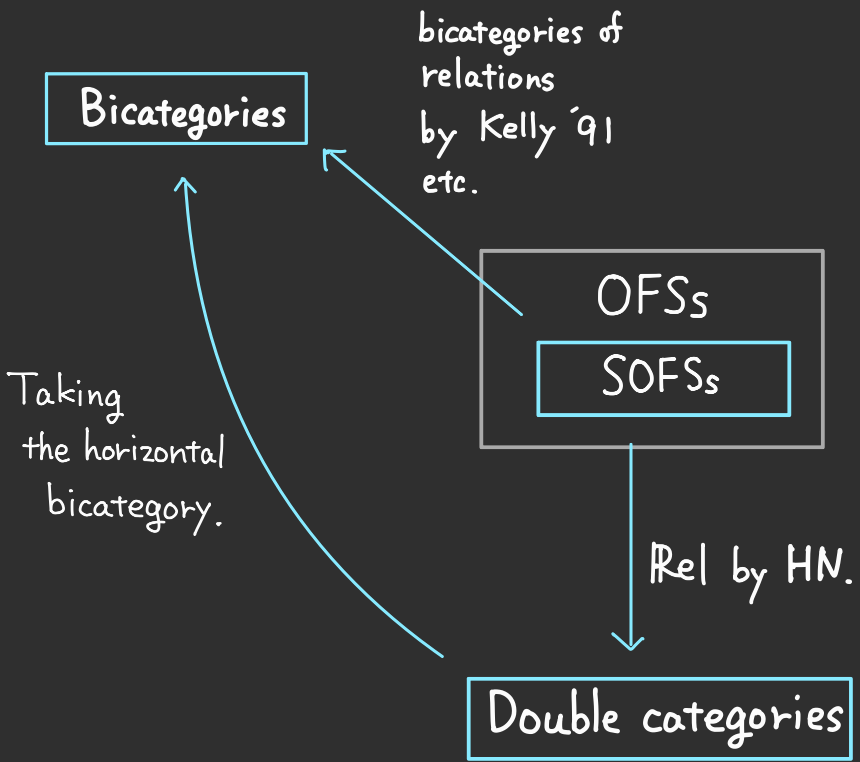

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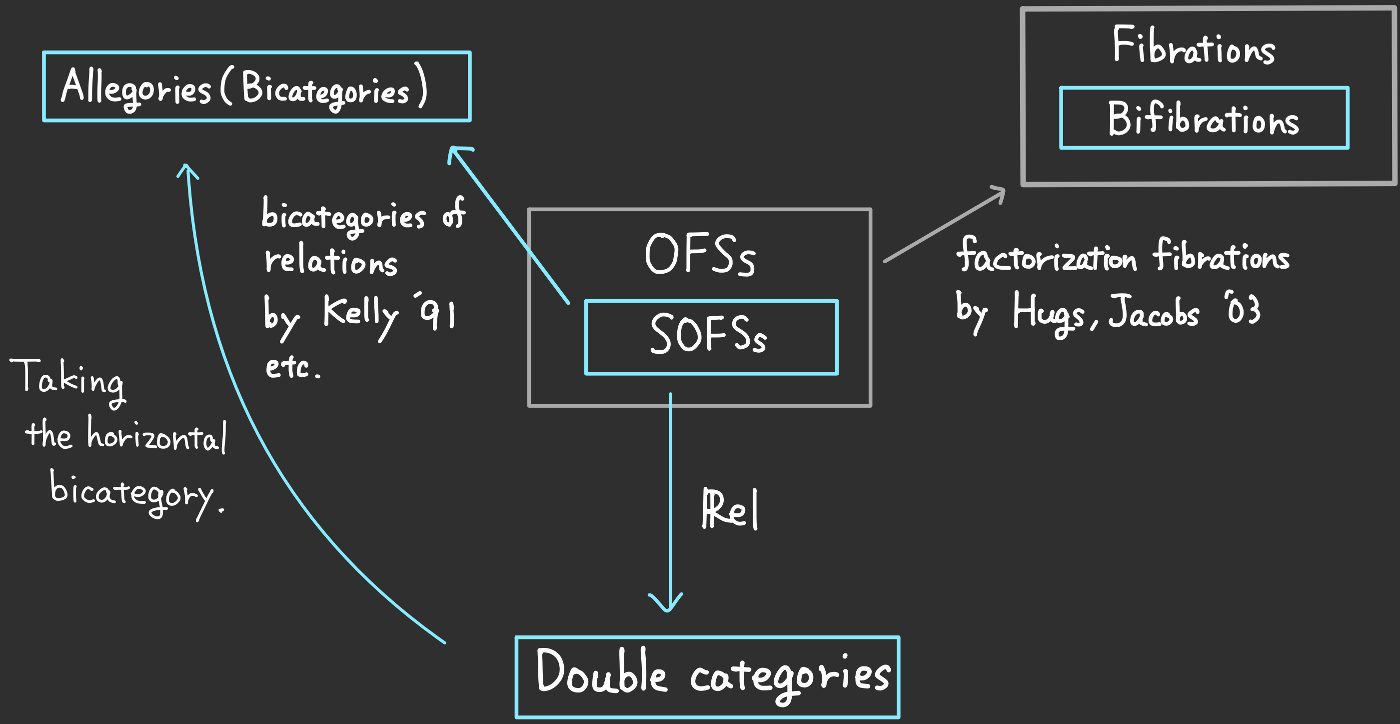

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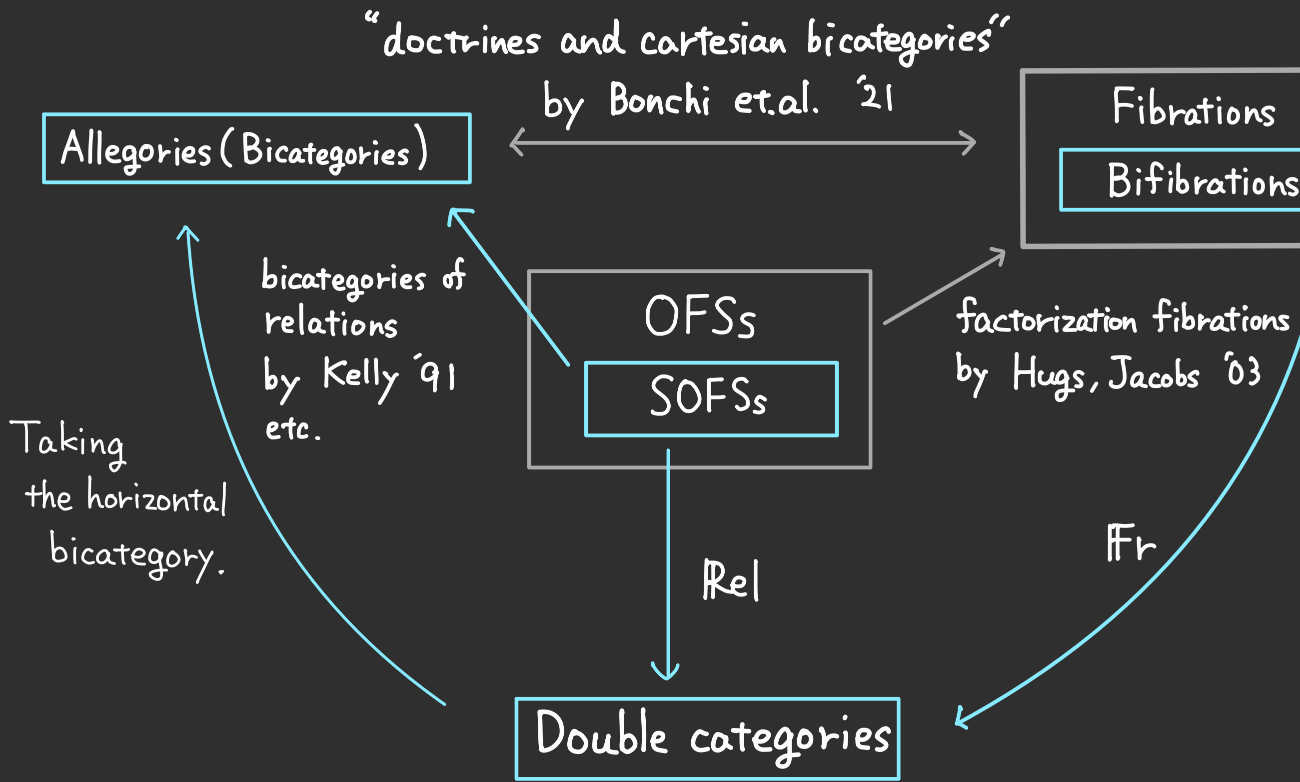

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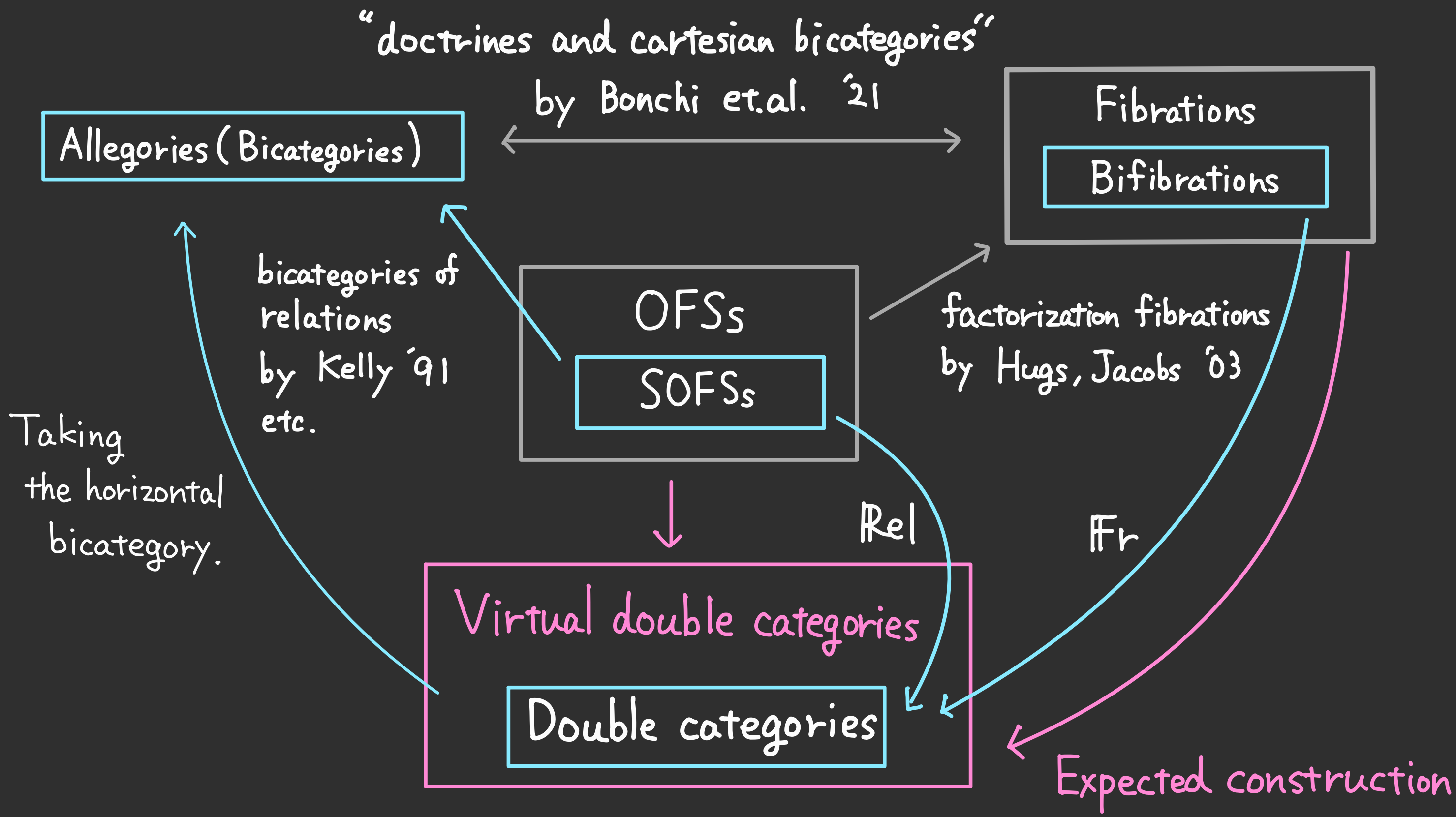


Classes of fibrations

A fibration $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is called **primary** if $\mathcal{B}$ and all the fibers $\mathcal{E}_I$ have finite products and all the reindexings $\mathcal{E}_J \xrightarrow{u^*} \mathcal{E}_I$ preserve them.

A primary fibration is called **existential** if the reindexings $\mathcal{E}_J \xrightarrow{\pi_J^*} \mathcal{E}_{I\times J}$ for all I and J have a left adjoint and they satisfy **the Beck-Chevalley condition** and **the Frobenius reciprocity**.

$$\exists i.\ \varphi(i,j) \longleftarrow\!\shortmid\ \varphi(i,j)$$

$$\mathcal{E}_J \underset{\pi_J^*}{\overset{\exists}{\rightleftarrows}} \mathcal{E}_{I\times J} \quad (\bot)$$

$$\psi(j) \longmapsto \psi(j)$$

$$\frac{i:I,\ j:J \vdash \varphi(i,j) \Rightarrow \psi(j)}{\overline{\ j:J \vdash \exists i:I.\ \varphi(i,j) \Rightarrow \psi(j)\ }}$$

Classes of fibrations

The Beck-Chevalley condition for existential fibrations :

$$\begin{array}{ccc} \mathcal{E}_{I\times J} & \xrightarrow{(id_I\times u)^*} & \mathcal{E}_{I\times K} \\ {\scriptstyle \exists_{I,J}}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle \exists_{I,K}} \\ \mathcal{E}_J & \xrightarrow[u^*]{} & \mathcal{E}_K \end{array}$$

given via the mate construction is an isomorphism.

The Frobenius reciprocity for existential fibrations :

$$\begin{array}{ccccc} \mathcal{E}_{I\times J}\times\mathcal{E}_J & \xrightarrow{id\times\pi_J^*} & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times J} & \xrightarrow{\wedge} & \mathcal{E}_{I\times J} \\ {\scriptstyle \exists_{I,J}\times id}\downarrow & & \swarrow\!\!\!\!\swarrow & & \downarrow{\scriptstyle \exists_{I,J}} \\ \mathcal{E}_J\times\mathcal{E}_J & & \xrightarrow[\wedge]{\hspace{6em}} & & \mathcal{E}_J \end{array}$$

given via the mate construction is an isomorphism.

Example : For an OFS (E, M) on a finitely complete category $\mathcal{C}$

$\mathcal{C}^{\to} \supseteq M \xrightarrow{cod} \mathcal{C}$ is a primary fibration.

If (E, M) is stable, then it is elementary and existential.

Classes of fibrations

$$\mathcal{E}_J \underset{\pi_J^*}{\overset{\exists_{I,J}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}_{I\times J} \quad ; \quad \exists_{I,J}\Big(\varphi(i:I,\, j:J)\Big) \text{``}\equiv\text{''}\ \exists i:I.\ \varphi(i,j)$$

The BC condition : $\exists i:I.\ \varphi(i, u(k)) \equiv (\exists i:I.\ \varphi(i,j))[u(k)/j]$

The Frobenius reciprocity : $\exists i:I.\big(\varphi(i,k) \wedge \psi(k)\big) \equiv \big(\exists i:I.\ \varphi(i,k)\big) \wedge \psi(k)$

Example : For an OFS (E, M) on a finitely complete category $\mathcal{C}$

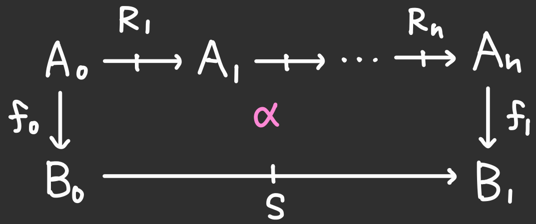
$\mathcal{C}^{\to} \supseteq \mathcal{M} \xrightarrow{\text{cod}} \mathcal{C}$ is a primary fibration.

If (E, M) is stable, then it is existential.

Virtual double categories

A virtual double category (VDC) $\mathbb{D}$ consists of

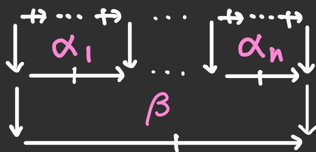
- objects
- vertical arrows
- horizontal arrows "without composition"
- virtual cells



← paths of finite length

with composition

$\beta\{\alpha_1, \dots, \alpha_n\}$:=



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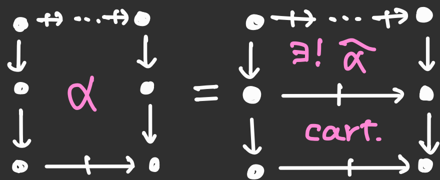
Restrictions and cartesianness are defined similarly to those for double categories.



A VDC $\mathbb{D}$ is called fibrant (FVDC) if any niche has its restriction.

From fibrations to VDCs

Construction

$$\begin{array}{c} \mathcal{E} \\ p \downarrow \\ \mathcal{B} \end{array} : \text{a primary fibration} \rightsquigarrow \text{a cartesian fibrant VDC } \mathbb{V}(p)$$

- The vertical part of $\mathbb{V}(p)$ is $\mathcal{B}$.
- The horizontal arrows $A \overset{R}{\to} B$ are the objects $R \in \mathcal{E}_{A\times B}$.
- The virtual cells

$$\begin{array}{ccccc} A_0 & \xrightarrow{R_1} & A_1 & \xrightarrow{R_2} & A_2 \\ f_0\downarrow & & \alpha & & \downarrow f_1 \\ B_0 & & \xrightarrow[S]{} & & B_1 \end{array}$$

are the arrows

$$\alpha : \pi_{01}^* R_1 \wedge \pi_{12}^* R_2 \longrightarrow S \quad \text{in } \mathcal{E}$$

above $A_0 \times A_1 \times A_2 \xrightarrow{\pi_{02}} A_0 \times A_1 \xrightarrow{f_0 \times f_1} B_0 \times B_1$ in $\mathcal{B}$.

Thm $\mathbb{V}$ gives a 2-functor $\mathbf{PrimFib} \longrightarrow \mathbf{FibVDbl}_{\mathbf{cart}}$

Example

An OFS (E, M) on a category with finite limits $\mathcal{C}$ gives a primary fibration $\begin{array}{c} M \\ \downarrow \text{cod} \\ \mathcal{C} \end{array}$.

$\mathbb{V}\left(\begin{array}{c} M \\ \downarrow \text{cod} \\ \mathcal{C} \end{array}\right)$			
	$\begin{array}{c} A \\ f \downarrow \\ B \end{array}$	$A \overset{R}{\nrightarrow} B$	$\begin{array}{ccccc} A_0 & \overset{R_1}{\nrightarrow} & A_1 & \overset{R_2}{\nrightarrow} & A_2 \\ f_0 \downarrow & & \alpha & & \downarrow f_1 \\ B_0 & & \underset{S}{\nrightarrow} & & B_1 \end{array}$
	$\begin{array}{c} A \\ f \downarrow \\ B \end{array}$ in $\mathcal{C}$	$\begin{array}{c} R \\ \langle l_R, r_R \rangle \downarrow \\ A \times B \end{array}$ in M	$\begin{array}{ccc} \pi_{01}^* R_1 \wedge \pi_{12}^* R_2 & \rightarrowtail & A_0 \times A_1 \times A_2 \\ \alpha \downarrow & \circlearrowleft & \downarrow \langle f_0, f_1 \rangle \\ S & \rightarrowtail & B_0 \times B_1 \end{array}$

By internal logic, α is interpreted as a Horn sequence :

$$x_0 : A_0, x_1 : A_1, x_2 : A_2 \mid R_1(x_0, x_1) \wedge R_2(x_1, x_2) \Rightarrow S(f_0(x_0), f_1(x_1))$$

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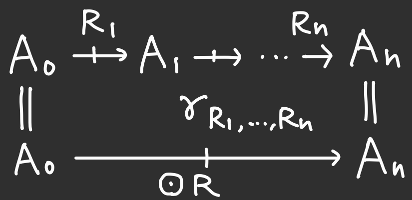
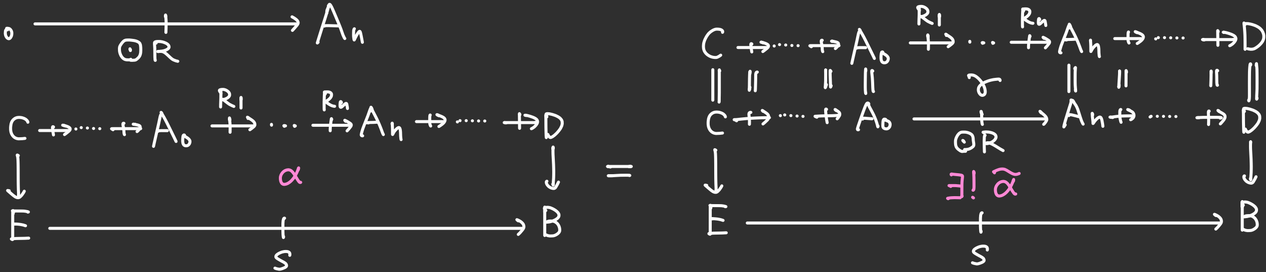
∃ in VDCs

A path of horizontal arrows $A_0 \overset{R_1}{\nrightarrow} A_1 \nrightarrow \cdots \overset{R_n}{\nrightarrow} A_n$ is composable if it comes equipped with a horizontal arrow $A_0 \xrightarrow{\odot R} A_n$ and a cell



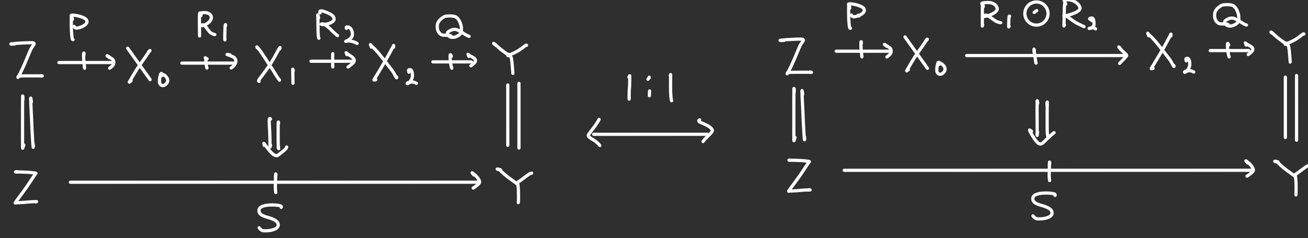
s.t.



This is a certain kind of double colimits.

∃ in VDCs

The universal property of the composite $R_1 \odot R_2$ can be seen as :



The logical interpretation is $\exists x_1.\, R_1(\bullet, x_1) \wedge R_2(x_1, \bullet)$.

$$\frac{P(z,x_0),\ R_1(x_0,x_1),\ R_2(x_1,x_2),\ Q(x_2,y) \Longrightarrow S(z,y)}{P(z,x_0),\ \exists x_1.\big(R_1(x_0,x_1) \wedge R_2(x_1,x_2)\big),\ Q(x_2,y) \Longrightarrow S(z,y)}$$

The composability of paths of positive length is

the double categorical counterpart of the existential quantifier.

$\exists$ in VDCs

Thm $\mathfrak{p}$ is existential iff $\mathbb{V}(\mathfrak{p})$ is composable.

$$\begin{array}{ccc} \mathsf{ExFib} & \longrightarrow & \mathsf{CFibVDbl}_{cart} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{PrimFib} & \xrightarrow[\mathbb{V}]{} & \mathsf{FibVDbl}_{cart} \end{array}$$

is a 2-pullback.

Logic	Fibrations	FVDCs
$\exists$	existential fibrations	cartesian composable FVDCs

∃ in VDCs

Thm p is existential iff $\mathbb{V}(p)$ is composable.

$$\begin{array}{ccc} \mathbf{ExFib} & \longrightarrow & \mathbf{CFibVDbl}_{\mathrm{cart}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{PrimFib} & \xrightarrow[\mathbb{V}]{} & \mathbf{FibVDbl}_{\mathrm{cart}} \end{array}$$

is a 2-pullback.

Logic	Fibrations	FVDCs
∃	existential fibrations	cartesian composable FVDCs
=	elementary fibrations	cartesian unital FVDCs

VDCs vs hyperdoctrines (Future work)

In terms of VDCs,

- some constructions using relational structures can be well handled. (e.g., quotient completions)
- quantifiers $\exists, \forall$ and equality $=$ are captured by double (co) limits, which are in the scope of formal category theory.

The core problem

It seems that the internal logic of VDCs is a proper extension of regular logic.

$$\exists x. (p(x) \wedge q(x))$$

$$1 \xrightarrow[\;\;+\;\;]{p(x)} A \xrightarrow[\;\;+\;\;]{q(x)} 1 \quad \neq \quad 1 \xrightarrow[\;\;+\;\;]{\top} A \xrightarrow[\;\;+\;\;]{p(x) \wedge q(x)} 1$$

Might be called directed logic?

Thank you!

hnasu@kurims.kyoto-u.ac.jp

Hayato Nasu

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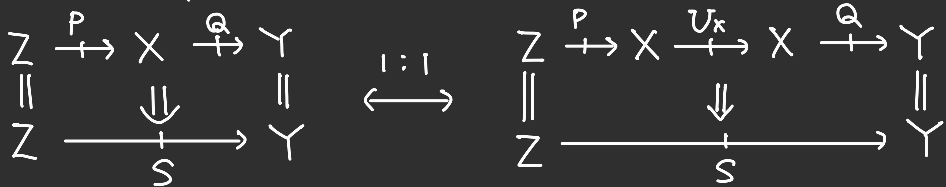
$=$ in VDC

- A primary fibration is called elementary if the reindexings $\mathcal{E}_{I\times I\times J} \xrightarrow{(\Delta_I \times id_J)^*} \mathcal{E}_{I\times J}$ for all I and J have left adjoints and they satisfy the Beck-Chevalley condition and the Frobenius reciprocity.

- The composite of a path of length 0 is called a unit.

- The universal property of the unit U_X is seen as :



$$\begin{array}{c} P(z,x),\ Q(x,y) \Rightarrow S(z,y) \\ \hline\hline P(z,x),\ x = x',\ Q(x',y) \Rightarrow S(z,y) \end{array}$$

- $$\begin{array}{ccc} \mathsf{ElemFib} & \longrightarrow & \mathsf{UFibVDbl}_{cart} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{PrimFib} & \underset{\simeq}{\longrightarrow} & \mathsf{FibVDbl}_{cart} \end{array}$$ is a 2-pullback.

Combining this with the 2-pullback in the previous slide,

we have $\mathsf{ElemExFib} \xrightarrow{\simeq} \mathsf{FibDbl}_{cart}$.

An equivalence

Restricting this to the full sub-2-categories, we have an equivalence :

$$\mathcal{E}\mathrm{lem}\mathcal{E}\mathrm{xFib}_{\mathrm{Frob}} \xrightarrow{\simeq} \mathcal{E}\mathrm{q}_{\mathrm{cart},\mathrm{Frob}}$$

$\mathcal{E}\mathrm{lem}\mathcal{E}\mathrm{xFib}_{\mathrm{Frob}}$ is the 2-category of fibrations satisfying the BC

for all the pullback of the form

$$\begin{array}{ccc} & I & \\ \swarrow & \lrcorner & \searrow \\ I^m & & I^n \\ \searrow & & \swarrow \\ & I^\ell & \end{array}$$

where

$$\begin{array}{ccc} & 1 & \\ \nearrow & \lrcorner & \nwarrow \\ m & & n \\ \nwarrow & & \nearrow \\ & \ell & \end{array}$$

in **FinSet** .