Relative simplicity via clans

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1. Generalized algebraic theory and clans

2. Simple clan morphisms

3. Application : regular decomposition number

Yuto Kawase and Hayato Nasu,
"On the decomposition of a strong epimorphism into regular epimorphisms," arXiv:2604.05744.

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Algebraic theory

Example The theory of monoids $\mathbb{T}\mathrm{mnd}$ (Single-sorted)

Sorts $\vdash *$

Operators $x : *,\ y : * \vdash \mu(x,y) : *\ ,\quad \vdash e : *$

Axioms $x : * \vdash \mu(e,x) = x : *\ ,\ \ldots$

Example The theory of pairs of a ring and its module $\mathbb{T}\mathrm{mdl}$

Sorts $\vdash \mathrm{Rng}\ ,\quad \vdash \mathrm{Md}$

Operators $x : \mathrm{Rng},\ y : \mathrm{Rng} \vdash \underbrace{\mathrm{sum}^{R}(x,y)}_{x+y} : \mathrm{Rng}\ ,\ \ldots$

$x : \mathrm{Rng},\ z : \mathrm{Md} \vdash \underbrace{\mathrm{act}(x,z)}_{x \cdot z} : \mathrm{Md}\ ,\ \ldots$

Axioms (Ring axioms) + (Module axioms)

Algebraic theory

<u>Example</u> The theory of monoids $\mathbb{T}\mathrm{mnd}$ (Single-sorted)

Sorts $\vdash *$

Operators $\underline{x:*,\ y:*} \vdash \mu(x,y) : *$, $\underline{\quad} \vdash e : *$

Axioms $\underline{x:*} \vdash \mu(e,x) = x : *$,

<u>Example</u> The theory of pairs of a ring and its module $\mathbb{T}\mathrm{mdl}$

Sorts $\vdash \mathrm{Rng}$, $\vdash \mathrm{Md}$

Operators $\underline{x : \mathrm{Rng},\ y : \mathrm{Rng}} \vdash \underset{x+y}{\mathrm{sum}^R(x,y)} : \mathrm{Rng}$

$\underline{x : \mathrm{Rng},\ z : \mathrm{Md}} \vdash \underset{x \cdot z}{\mathrm{act}(x,z)} : \mathrm{Md}$

Axioms (Ring axioms) + (Module axioms)

Contexts
$\parallel$
Variable declaration

Models of a theory

A model of an algebraic theory $\mathbb{T}$ (in Set) consists of :

(1) $\vdash \sigma$ (2) $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash f(x_1, \dots, x_n) : \tau$

$\Downarrow$ $\Downarrow$

$M_\sigma \in \mathbf{Set}$ $M_{\sigma_1} \times \cdots \times M_{\sigma_n} \xrightarrow{M_f} M_\tau$ in Set

s.t. the axioms in $\mathbb{T}$ are satisfied.

A morphism between models is a family of function $M_\sigma \xrightarrow{\varphi_\sigma} N_\sigma$ that preserves operators.

$\mathrm{Mod}(\mathbb{T})$: the category of models of $\mathbb{T}$

(Multisorted) Lawvere theory

$\mathbb{T}$: algebraic theory $\rightsquigarrow$ $\mathcal{C}\ell(\mathbb{T})$: a category w/ fin. products

Objects : contexts $(x_1 : A_1, \ldots, x_n : A_n)$

Morphisms : equiv. classes of sequences of terms

$$\frac{(x_1 : A_1, \ldots, x_m : A_m) \xrightarrow{[t_1, \ldots, t_n]} (y_1 : B_1, \ldots, y_n : B_n)}{x_1 : A_1, \ldots, x_m : A_m \vdash t_i : B_i \quad \text{modulo provable equality in } \mathbb{T}}$$

Example $\mathcal{C}\ell(\mathbb{T}_{\mathrm{mnd}})$

$$\begin{tikzcd}
(x : *, y : *) \arrow[r, bend left, "{[x \cdot y]}"] \arrow[r, bend right, "{[x \cdot (e \cdot y)]}"'] \arrow[r, phantom, "\|"] & (z : *)
\end{tikzcd}$$

(Multisorted) Lawvere theory (ctn.)

$\mathbb{T}$: algebraic theory

$\rightsquigarrow$ $\mathcal{C}_{\mathbb{T}}$: a category w/ fin. products

Objects : contexts $(x_1 : A_1, \dots, x_n : A_n)$

Morphisms : equiv. classes of sequences of terms

$$\frac{(x_1 : A_1, \dots, x_m : A_m) \xrightarrow{[t_1, \dots, t_n]} (y_1 : B_1, \dots, y_n : B_n)}{x_1 : A_1, \dots, x_m : A_m \vdash t_i : B_i} \text{ modulo provable equality in } \mathbb{T}$$

Terminals : the empty context $(\,)$

Binary products : $(x_1 : A_1, \dots, x_n : A_n) \times (y_1 : B_1, \dots, y_m : B_m)$

$$= (x_1 : A_1, \dots, x_n : A_n, y_1 : B_1, \dots, y_m : B_m)$$

(Check!)

<u>Proposition</u> $\mathrm{Mod}(\mathbb{T}) \simeq \mathbf{FinProd}(\mathcal{C}\ell(\mathbb{T}), \mathbf{Set})$

The category of finite product preserving functors $\mathcal{C}_{\mathbb{T}} \to \mathbf{Set}$

Generalized algebraic theory (GAT)

||

[Cartmell '78]

Algebraic theory with dependent sorts

Example The theory of categories $\mathbb{T}_{\mathrm{cat}}$

Sorts

$$\vdash \mathrm{Ob}$$

$$x : \mathrm{Ob},\ y : \mathrm{Ob} \vdash \mathrm{Mor}(x, y)$$

Operators

$$x : \mathrm{Ob} \vdash \mathrm{id}(x) : \mathrm{Mor}(x, x)$$

$$x, y, z : \mathrm{Ob},\ f : \mathrm{Mor}(x, y),\ g : \mathrm{Mor}(y, z) \vdash \underbrace{\mathrm{comp}(x, y, z, f, g)}_{g \circ f} : \mathrm{Mor}(x, z)$$

Axioms

$$x : \mathrm{Ob},\ f : \mathrm{Mor}(x, y) \vdash \underbrace{\mathrm{comp}(x, x, y, \mathrm{id}(x), f) = f}_{f \circ \mathrm{id}_x = f} : \mathrm{Mor}(x, y)$$

...

The syntactic category of a GAT

$\mathbb{T}$: GAT $\rightsquigarrow$ $\mathcal{Cl}(\mathbb{T})$: a category

Objects : contexts $(x_1 : A_1,\ x_2 : A_2(x_1), \ldots,\ x_m : A_m(x_1, \ldots, x_{m-1}))$

Morphisms : equiv. classes of sequences of terms

$$\frac{(x_1 : A_1, \ldots, x_m : A_m(x_1, \ldots, x_{m-1})) \xrightarrow{[t_1, \ldots, t_n]} (y_1 : B_1, \ldots, y_n : B_n(y_1, \ldots, y_{n-1}))}{x_1 : A_1, \ldots, x_m : A_m(x_1, \ldots, x_{m-1}) \vdash t_i : B_i(t_1, \ldots, t_{i-1})}$$

modulo provable equality in $\mathbb{T}$

This category has finite products by the same argument for ATs.

The syntactic category of a GAT (ctn.)

Definition. A map in $\mathcal{Cl}(\mathbb{T})$ that is isomorphic to

$$(x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1})) \xrightarrow{p_i} (x_1 : A_1, \ldots, x_i : A_i(x_1, \ldots, x_{i-1}))$$

is called **a display map** (= a fibration).

Proposition. $\mathcal{Cl}(\mathbb{T})$ admits a pullback of a display map along a map, which is again a display map.

$\boxed{\text{Idea}}$

$$\begin{tikzcd}
(y_1 : B_1, x_3 : A_3(t_1, t_2)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (x_1 : A_1, x_2 : A_2(x_1), x_3 : A_3(x_1, x_2)) \arrow[d] \\
(y_1 : B_1) \arrow[r, "{[t_1, t_2]}"'] & (x_1 : A_1, x_2 : A_2(x_1))
\end{tikzcd}$$

Clans

Definition [Joyal '17]

Display maps

A clan $(\mathcal{C}, \Delta)$ is a category $\mathcal{C}$ with a class $\Delta \subseteq \mathrm{Mor}(\mathcal{C})$

s.t. (1) $A \xrightarrow{\cong} A' \in \Delta$, (2) $A \xrightarrow{f} B,\ B \xrightarrow{g} C \in \Delta \Rightarrow A \xrightarrow{gf} C \in \Delta$

(3) $\exists\ () \in \mathcal{C}$: terminal (4) $^\forall X \in \mathcal{C} \quad X \xrightarrow{!} () \in \Delta$

(5)

$$\exists \quad \begin{tikzcd} P \arrow[r] \arrow[d, "f^*p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "p \in \Delta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

(6) $f^*p \in \Delta$

Corollary $\mathcal{C}\ell(\mathbb{T})$ with displayed maps forms a clan.

Clans (ctn.)

① finite-product clan

$\mathcal{C}$: a caty w/ fin. products

$\rightsquigarrow (\mathcal{C}, \{\text{product projections}\})$ is a clan.

② finite-limit clan

$\mathcal{C}$: a caty w/ fin. limits

$\rightsquigarrow (\mathcal{C}, \{\text{all maps}\})$ is a clan.

In particular,

$\mathbf{Set} := (\mathrm{Set}, \mathrm{All})$ is a clan.

Definition.

A clan morphisms $(\mathcal{C}, \Delta) \to (\mathcal{D}, \Theta)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ that preserves display maps, the terminal, and the pullbacks.

Definition

$$\mathrm{Mod}(\mathbb{T}) := \mathrm{CLAN}(\mathcal{C}\ell(\mathbb{T}), \mathbf{Set})$$

Functorial semantics

Syntactic theory	Algebraic theory	$\subseteq$	Generalized algebraic theory
Sorted theory	(Multisorted) Lawvere theory	$\subseteq$	Contextual categories, ... *1
Sort-free theory	Categories w/ finite products	$\subseteq$	Clans
Category of models	locally strongly finitely presentable	$\subseteq$	locally finitely presentable

*1 There are a plenty of categorical frameworks for dependent type theory with subtle differences.

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Simplcity itself!

Slogan!

An algebraic theory is a simple* generalized algebraic theory

It is a GAT with no sort that is dependent on others.

$\Longrightarrow$ The category of models has a special property.

- It is monadic over $\mathbf{Set}^I$
- It is regular.

An example of non-simple GATs is $\mathbb{T}_{cat}$. (**Cat** is not regular!)

✓ $\vdash \mathrm{Ob}$ ✗ $x : \mathrm{Ob},\ y : \mathrm{Ob} \vdash \mathrm{Mor}(x, y)$

* The term is from simply-typed lambda calculus.

Relative simplicity

✓ $\vdash \mathrm{Ob}$

✗ $x : \mathrm{Ob},\ y : \mathrm{Ob} \vdash \mathrm{Mor}(x, y)$

Mor is simple
if Ob is on the baseline.

Slogan 2 Simplicity is just a relative notion.

Definition. $\mathbb{T}$ is simple over $\mathbb{T}' \subseteq \mathbb{T}$

if for every sort symbol $\Gamma \vdash A$ in $\mathbb{T}$, Γ is in $\mathbb{T}'$

Example $\mathbb{T}_{set}$ has only one sort Ob and no operator and axiom.

① $\mathbb{T}_{set}$ is simple (over the empty theory)

② $\mathbb{T}_{cat}$ is simple over $\mathbb{T}_{set}$

$\leftrightsquigarrow$ The theory of categories with fixed set of objects is simple.

Definition. A clan $(\mathcal{C},\Delta)$ is generated by a class $\Delta' \subseteq \Delta$ if Δ is the smallest clan structure on $\mathcal{C}$ containing Δ'.

Proposition. [KN.] $(\mathcal{C},\Delta)$ is generated by Δ' if

$$\Delta = \left\{ X_0 \xrightarrow{\varphi_1} X_1 \to \cdots \xrightarrow{\varphi_n} X_n \;\middle|\; \begin{tikzcd} X_i \arrow[r] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\lrcorner"] & Y_i \arrow[d] \\ X_{i+1} \arrow[r] & Y_{i+1} \end{tikzcd} \in \Delta' \cup \mathrm{Iso} \cup \{X \xrightarrow{!} 1\} \right\}$$

Example The clan $\mathrm{Cl}(\mathbb{T})$ is generated by

$$\left\{ (\Gamma, y : B(x_1,\dots,x_n)) \xrightarrow{p_n} (\Gamma) \;\middle|\; \Gamma \vdash B(x_1,\dots,x_n) \text{ sort symbol} \right\}$$

For $\mathbb{T}_{cat}$, $\{(x:\mathrm{Ob}) \to (),\ (x,y:\mathrm{Ob},\ f:\mathrm{Mor}(x,y)) \to (x,y:\mathrm{Ob})\}$

Definition. A clan morphism $\Phi : (\mathcal{C}, \Delta) \longrightarrow (\mathcal{D}, \Theta)$ is simple if $(\mathcal{D}, \Theta)$ is generated by $\{Y \to \Phi(X) \in \Theta\}_{X \in \mathcal{C}, Y \in \mathcal{D}}$

Example

The empty theory

- The clan morphism $1 \longrightarrow (\mathcal{C}, \Delta)$; $() \longmapsto ()$ is simple iff $(\mathcal{C}, \Delta)$ is the finite-product clan $\simeq$ algebraic theory.

- We have a simple clan morphism $\mathrm{Cl}(\mathbb{T}') \longrightarrow \mathrm{Cl}(\mathbb{T})$ when $\mathbb{T}$ is simple over $\mathbb{T}' \subseteq \mathbb{T}$.

  In particular, the following is simple.

  $\mathrm{Cl}(\mathbb{T}_{\mathrm{set}}) \longrightarrow \mathrm{Cl}(\mathbb{T}_{\mathrm{cat}})$; $(x_1 : \mathrm{Ob}, \dots, x_n : \mathrm{Ob}) \mapsto (x_1 : \mathrm{Ob}, \dots, x_n : \mathrm{Ob})$

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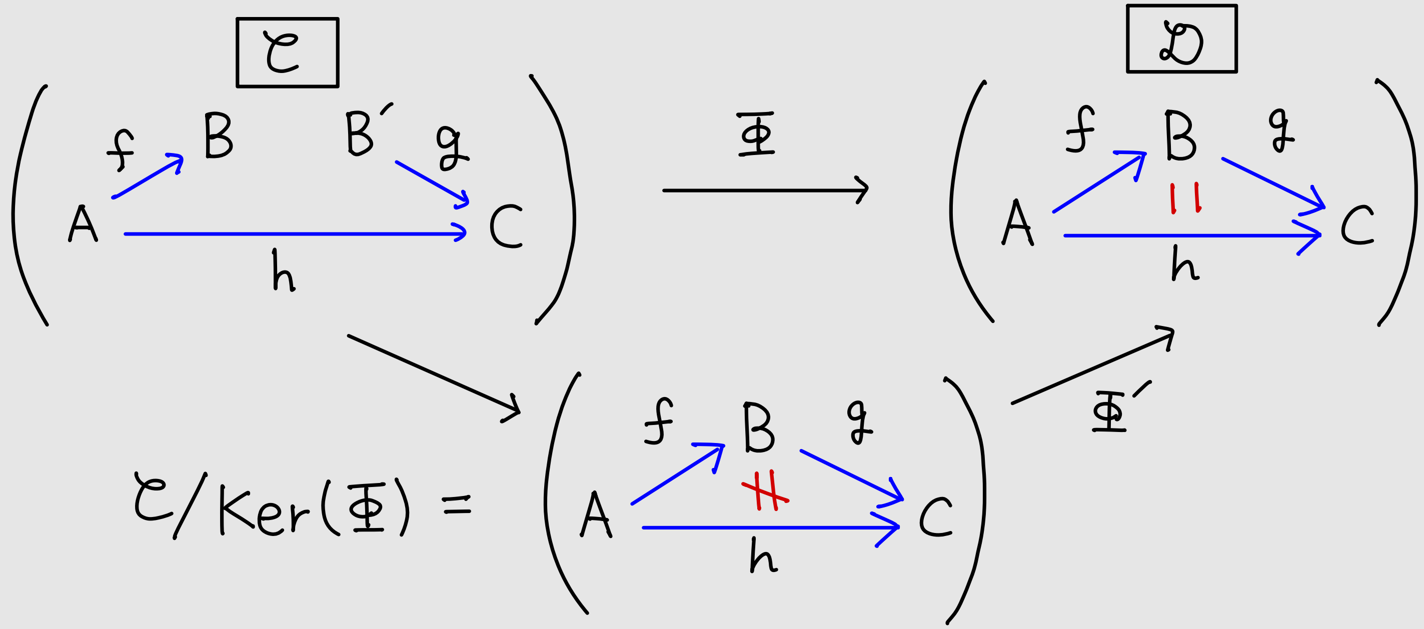
Application: regular epimorphism decomposition

In a regular category, strong epimorphisms = regular epimorphisms

$$\Longleftrightarrow \quad \mathrm{Im}(f) \cong A/\mathrm{Ker}(f)$$

In **Cat**, strong epimorphisms = composites of 2 regular epimorphisms



Question. How many regular epimorphisms are needed in $\mathrm{Mod}(\pi)$?

Simplicity and regular decomposition

Proposition*. [KN.]

$\mathscr{C}' \underset{i}{\overset{\perp}{\rightleftarrows}} \mathscr{C}$ in CLAN, i : simple

If strong epimorphisms = composites of n regular epimorphisms in $\mathrm{Mod}(\mathscr{C}')$,

then

strong epimorphisms = composites of $(n+1)$ regular epimorphisms in $\mathrm{Mod}(\mathscr{C})$.

Example

$$\left(1 \underset{\text{simple}}{\overset{\perp}{\rightleftarrows}}\right) \mathrm{Cl}(\mathbb{T}_{\mathrm{set}}) \underset{\text{simple}}{\overset{\perp}{\rightleftarrows}} \mathrm{Cl}(\mathbb{T}_{\mathrm{cat}})$$

* The precise statement involves the regular decomposition number.

Dependency rank

Definition.

> In a GAT $\mathbb{T}$, we inductively define **the dependency rank** $dr(A)$ for each well-formed type $\Gamma \vdash A$ by
>
> $$dr(A) = \max\{dr(B) \mid B \text{ appears in } \Gamma\} + 1.$$
>
> The height in the tree of sorts

Example.

In $\mathbb{T}_{cat}$, $\quad dr(Ob) = 1$, $dr(Mor(x,y)) = 2$

$$\begin{tikzcd} Mor \ ② \arrow[d, dash] \\ Ob \ ① \end{tikzcd}$$

Measuring regular decomposition number

We call $\mathbb{T}$ a non-descending if

- $\Gamma \vdash f(\vec{x}) : B$: operator symbol
- $\Gamma \vdash t = s : B$: axiom

$\Longrightarrow \quad dr(B) \geq \max_{A \text{ in } \Gamma} dr(A)$

When $\mathbb{T}$ is non-descending and $\max\{dr(A) \mid A \text{ in } \mathbb{T}\} = n$

we have a chain of GATs :

The subtheory spanned by types with $dr \leq i$

$$\emptyset \subseteq \mathbb{T}_{dr\leq 0} \subseteq \cdots \subseteq \mathbb{T}_{dr\leq i} \subseteq \cdots \subseteq \mathbb{T}_{dr\leq n} = \mathbb{T}$$

This induces

$$1 \underset{\text{simple}}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}} \mathrm{Cl}(\mathbb{T}_{dr\leq 0}) \underset{\text{simple}}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}} \cdots \underset{\text{simple}}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}} \mathrm{Cl}(\mathbb{T})$$

Main result

Theorem. [KN.]

> $\mathbb{T}$: non-descending, $\max\{ \mathrm{dr}(A) \mid A \text{ in } \mathbb{T}\} = n$
> $\Rightarrow$ In $\mathrm{Mod}(\mathbb{T})$,
> strong epimorphisms = composites of $(n+1)$ regular epimorphisms.

Example

nCat	strong epimorphisms	=	composites of	$(n+1)$	regular epimorphisms.
strMonCat			"	2	"
strDblCat			"	3	"

Thank you !

hayatonasu.github.io

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arxiv:2604.05744

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Conclusion

A1/A3

Summary

- Simplicity ($\doteqdot$ no iterated dependency) is a relative notion.
- Relative simplicity is expressed as a property of clan morphisms.
- Simplicty of clan morphisms can imply categorical properties of categories of models (like regularity)

Future directions

- How much of GAT/dependent type theory can be developed purely with clans/tribes.

Relative theory

Proposition. [KN.]

$\mathscr{C}' \underset{i}{\overset{\perp}{\rightleftarrows}} \mathscr{C}$: reflective full subclan, i : simple,

N : a model of $\mathscr{C}'$,

$\Rightarrow$ The category $i^* \downarrow^{\cong} N$ is of the form $\mathrm{Mod}(\mathscr{C}[N])$

for a finite-product clan $\mathscr{C}[N]$. In particular, it is regular.

Example

$\mathscr{C}_{\Pi\mathrm{set}} \overset{\perp}{\rightleftarrows} \mathscr{C}_{\Pi\mathrm{cat}}$ is simple.

A model of $\Pi\mathrm{set}$ is just a set.

This is the theory of monoidoids $/N$

Then, $i^* \downarrow^{\cong} N \simeq (i^*)^{-1}(N) \simeq$ "$\{$categories A with $\mathrm{Ob}(A) = N\}$"

Decomposition number

Definition. [Gabriel, Ulmer '71, Börger '91, KN.]

The regular (canonical) decomposition number $\delta(f)$ of $A \xrightarrow{f} B$ is the smallest α at which the seq. stabilizes ($A_{\alpha,\alpha+1}$: iso).

$$
\begin{tikzcd}
A \arrow[r, two heads, "A_{0,1}"] \arrow[drrrr, bend right, "f"'] & A_1 \arrow[r, two heads, "A_{1,2}"] \arrow[drrr, bend right, "f_1"] & A_2 \arrow[r, two heads, "A_{2,3}"] \arrow[drr, bend right, "f_2"] & A_3 & \cdots \\
 & & & \cdots & B
\end{tikzcd}
$$

The global regular (canonical) decomposition number $\delta(\mathcal{C})$ of a category $\mathcal{C}$ is the smallest γ s.t. $\delta(f) < \gamma$ for any f in $\mathcal{C}$.