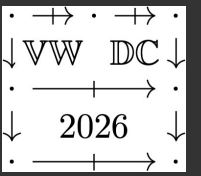
VIRTUAL DOUBLE CATEGORIES OF RELATIONS RELATIVE TO FIBRATIONS

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Third Virtual Workshop on Double Categories



Nov. 10th

$$\mathbb{S}\mathrm{pan} = \mathbb{B}\mathrm{il}\begin{pmatrix} \mathrm{Set}^{\rightarrow} \\ \downarrow \\ \mathrm{Set} \end{pmatrix} \qquad \mathbb{R}\mathrm{el}(\mathcal{C}) = \mathbb{B}\mathrm{il}\begin{pmatrix} \mathcal{C}^{\rightarrow} \\ \downarrow \\ \mathcal{C} \end{pmatrix}$$

$$\mathbb{R}\mathrm{el} = \mathbb{B}\mathrm{il}\begin{pmatrix} \mathrm{SubSet} \\ \downarrow \\ \mathrm{Set} \end{pmatrix}$$

$$\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C}) = \mathbb{B}\mathrm{il}\begin{pmatrix} \mathcal{C}^{\rightarrow|M} \\ \downarrow \\ \mathcal{C} \end{pmatrix} \qquad \mathbb{Q}\text{-}\mathbb{R}\mathrm{el} = \mathbb{B}\mathrm{il}\begin{pmatrix} \mathrm{Fam}(\mathbb{Q}) \\ \downarrow \\ \mathrm{Set} \end{pmatrix}$$

$\mathbb{B}$il stands for "bilateral."

1. The construction $\mathbb{B}\mathrm{il}$

2. When is $\mathbb{B}\mathrm{il}(\mathbb{P})$ a cartesian equipment?

3. When does an equipment arise from $\mathbb{B}\mathrm{il}$?

4. Double categorical logic

5. Ongoing work

1. The construction $\mathbb{B}\mathrm{il}$

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$\mathbb{R}\mathrm{el}$

objects A : sets

tight morphisms $A \xrightarrow{f} B$: functions

loose morphisms $A \overset{R}{\nrightarrow} B$: (binary) relations

cells

$$\begin{array}{ccc} A & \overset{R}{\nrightarrow} & B \\ f \downarrow & \Downarrow & \downarrow g \\ C & \underset{S}{\nrightarrow} & D \end{array} \quad : \quad \begin{array}{l} \forall x \in A.\ \forall y \in B. \\ \quad R(x, y) \Longrightarrow S(f(x), g(y)) \end{array}$$

Rel

$$A \xrightarrow{R} B \quad \odot \quad B \xrightarrow{S} C \quad = \quad A \xrightarrow{R \odot S} C$$

$$(R \odot S)(a, c) \overset{\text{def}}{\iff} \exists\, b \in B.\ R(a, b) \wedge S(b, c)$$

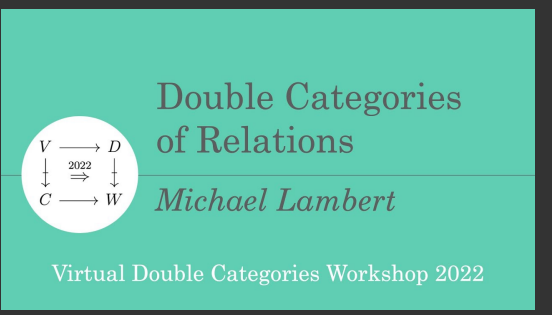
$$A \xrightarrow{Id_A} A$$

$$Id_A(a, a') \overset{\text{def}}{\iff} a = a'$$

Rel is a cartesian equipment.

$\mathbb{R}\mathrm{el}(\mathcal{C})$

$\mathcal{C}$: regular cat. [Lambert '22]



objects A : objects in $\mathcal{C}$

tight morphisms $A \xrightarrow{f} B$: morphisms in $\mathcal{C}$

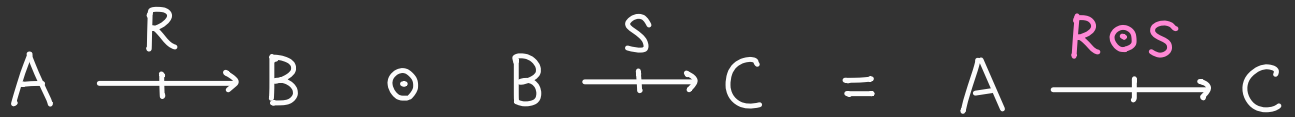
loose morphisms $A \xrightarrow{R} B$: $R \rightarrowtail A \times B$ in $\mathcal{C}$

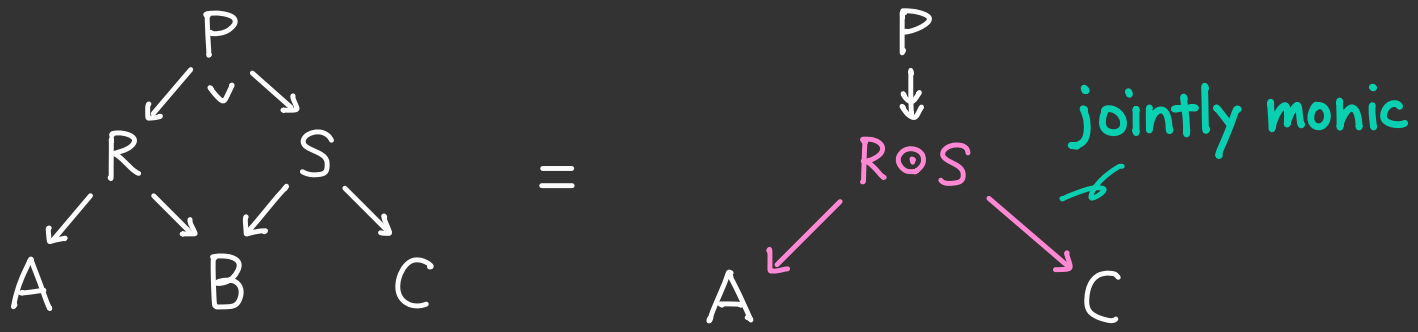
monomorphism

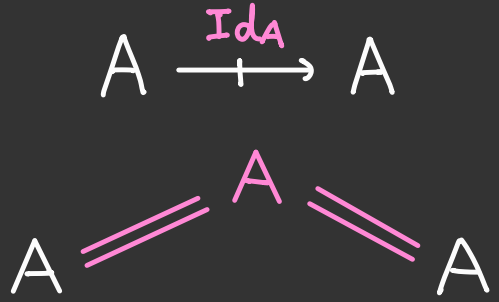
cells

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ {\scriptstyle f}\downarrow & \cong & \downarrow{\scriptstyle g} \\ C & \xrightarrow[S]{} & D \end{array} \quad : \quad \begin{array}{ccc} R & \rightarrowtail & A \times B \\ \downarrow & \cong & \downarrow {\scriptstyle f \times g} \\ S & \rightarrowtail & C \times D \end{array} \quad \text{in } \mathcal{C}$$

Rel(ℰ)







$\mathbb{R}\mathrm{el}_{(\mathcal{E},\mathcal{M})}(\mathcal{C})$ $\mathcal{C}$: cat. w/ fin. lim.
$(\mathcal{E},\mathcal{M})$: Stable OFS

objects A : objects in $\mathcal{C}$

tight morphisms $A \xrightarrow{f} B$: morphisms in $\mathcal{C}$

loose morphisms $A \xrightarrow{R} B$: $R \xrightarrow{c_R} A \times B$ in $\mathcal{M}$

cells

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ {\scriptstyle f}\downarrow & \varphi & \downarrow{\scriptstyle g} \\ C & \xrightarrow[S]{} & D \end{array} \quad : \quad \begin{array}{ccc} R & \xrightarrow{c_R} & A \times B \\ {\scriptstyle \varphi}\downarrow & \cong & \downarrow{\scriptstyle f \times g} \\ S & \xrightarrow[c_S]{} & C \times D \end{array} \quad \text{in } \mathcal{C}$$



$\mathcal{Q}$-Rel

$\mathcal{Q}$: cat. w/ fin. prod. & dist. coprod.

objects A : sets

tight morphisms $A \xrightarrow{f} B$: functions

loose morphisms $A \overset{R}{\nrightarrow} B$: $\big(R(a,b)\big)_{a \in A,\, b \in B}$, $R(a,b) \in \mathcal{Q}$

cells

$$\begin{array}{ccc} A & \overset{R}{\nrightarrow} & B \\ {\scriptstyle f}\downarrow & \varphi & \downarrow{\scriptstyle g} \\ C & \underset{S}{\nrightarrow} & D \end{array} \quad : \quad \left(\begin{array}{c} R(a,b) \\ \downarrow \varphi_{a,b} \\ S(f(a), g(b)) \end{array} \right)_{\substack{a \in A \\ b \in B}}$$

Binary products
+
Unary relations
$\rightsquigarrow$ Binary relations

$\wedge, \exists \rightsquigarrow$ Composition of relations

$\top, = \rightsquigarrow$ Identity relation

Logical system with $\wedge, \top, \exists, =$ $\rightsquigarrow$ Double category of relations

What is a logical system with $\wedge, \top, \exists, =$?

Answer 0 Regular logic

(A fragment of first-order logic with $\wedge, \top, \exists, =$.)

Answer 1 [Lawvere '69]

Logical systems = Hyperdoctrines

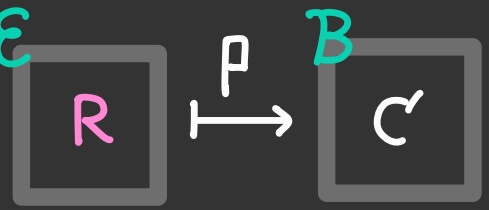
(elementary existential doctrines)

$P : \mathcal{C}^{op} \longrightarrow Pos$

w/ some structure

Answer 2 [Ehrahd '88, Pavlovic '96, Jacobs '99]

Logical systems = Fibrations $p : \mathcal{E} \to \mathcal{B}$

	Set theory	Fibration $\mathcal{E} \to \mathcal{B}$
Sets	$C \in \mathrm{Set}$	$C \in \mathcal{B}$
Functions	$C \xrightarrow{f} D \in \mathrm{Set}$	$C \xrightarrow{f} D \in \mathcal{B}$
Subsets ≅ Predicates	$R \subseteq C$ $R(x) \quad (x \in C)$	
Substitution	$\mathcal{E}_C \xleftarrow{f^*} \mathcal{E}_D$	$R(f(x)) \longleftarrow R(y)$
Existential quantifier $\exists$	$P(C \times D) \underset{\pi^*}{\overset{\exists_\pi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} P(C)$	$R(x,y) \mapsto \exists y.\, R(x,y)$

Answer 2 [Ehrahd '88, Pavlovic '96, Jacobs '99]

Logical systems = Fibrations

$p : \mathcal{E} \to \mathcal{B}$

Regular logic

$\wedge, \top, =, \exists$

$\leftrightsquigarrow$

Elementary existential fibrations

$\mathbb{B}\mathrm{il}(p)$ $\quad p : \mathcal{E} \to \mathcal{B} \in \mathrm{Fib}_{=,\exists}$ ← The class of elem. exist. fib.

cf. Ff in [Shulman '08]

objects A : objects in $\mathcal{B}$

tight morphisms $A \xrightarrow{f} B$: morphisms in $\mathcal{B}$

loose morphisms $A \xrightarrow{R} B$: $\quad R \overset{p}{\longmapsto} A \times B$

cells

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ {\scriptstyle f}\downarrow & \varphi & \downarrow{\scriptstyle g} \\ C & \xrightarrow[S]{} & D \end{array} :$$

$$\begin{array}{ccc} R & \longmapsto & A \times B \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle f \times g} \\ S & \longmapsto & C \times D \end{array}$$

$\mathbb{B}\mathrm{il}(p)$ $\quad p : \mathcal{E} \to B \in \mathrm{Fib}_{=,\exists}$

$$A \xrightarrow{R} B \;\circ\; B \xrightarrow{S} C \;=\; A \xrightarrow{R \circ S} C$$

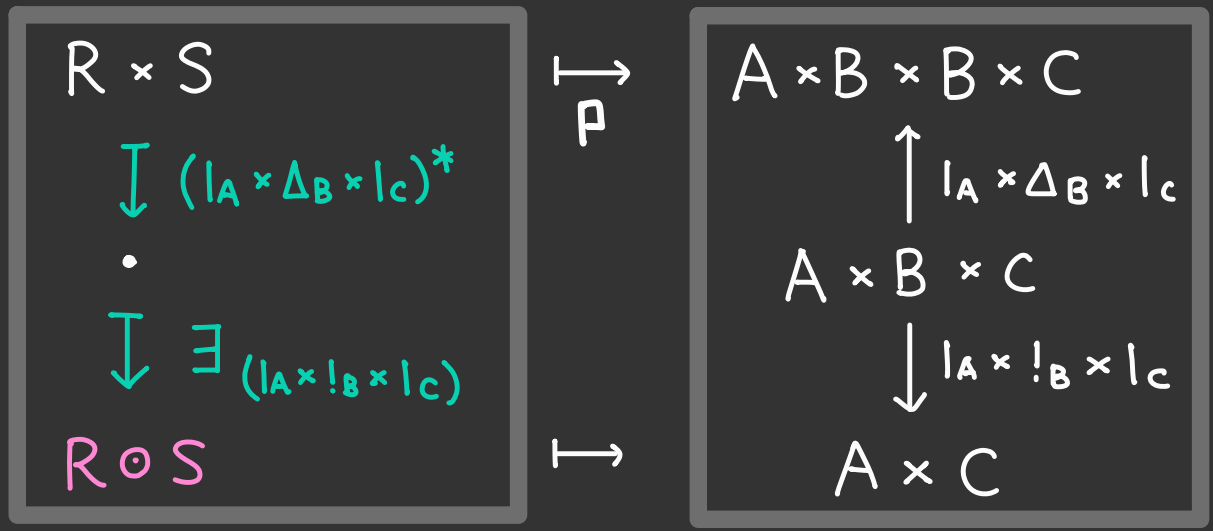
$R(a,b) \wedge S(b',c)$

$\downarrow$

$R(a,b) \wedge S(b,c)$

$\downarrow$

$\exists b.\,(R(a,b) \wedge S(b,c))$



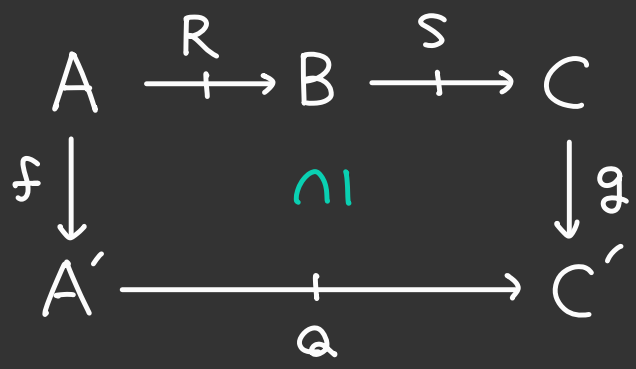
$$A \xrightarrow{Id_A} A \;=\; \exists_{\Delta_A}(\top_A)$$

What if we don't have $\exists$ & $=$?

- Non-regular categories.
- Non-stable OFSs.
- Non-elementary-existential fibrations.

Still, we can define virtual cells !

Binary



$$(\exists\, b \in B.\ R(a,b) \wedge S(b,c)) \longrightarrow Q(f(a), g(c)) \qquad (\forall a, c)$$

$$\Longleftrightarrow \quad R(a,b) \wedge S(b,c) \longrightarrow Q(f(a), g(c)) \qquad (\forall a, b, c)$$

only $\wedge$.

Nullary



$$\top \longrightarrow R(f(a), g(a)) \iff a = a' \longrightarrow R(f(a), g(a'))$$

$(\forall a)$ $\qquad$ $(\forall a, a')$

↑ Only $\top$.

Binary products
+
Unary relations
$\rightsquigarrow$
Binary relations
+ virtual cells

$\wedge, \exists \rightsquigarrow$ Composition of relations

$\top, = \rightsquigarrow$ Identity relations

Logical system
with $\wedge, \top, \exists, =$
$\rightsquigarrow$
Virtual double category
of relations

Definition A cartesian fibration is a cartesian object in the 2-cat. Fib.

$$P \underset{\perp}{\overset{\Delta}{\rightleftarrows}} P \times P \qquad P \underset{\perp}{\overset{!}{\rightleftarrows}} 1$$

$p : \mathcal{E} \longrightarrow \mathcal{B}$ in Fib is cartesian

$\Leftrightarrow$

(1) $\mathcal{B}$ has fin. products.

(2) Every fiber $\mathcal{E}_I$ has fin. products.

(3) Every base change preserves them.

Proposition [N.]

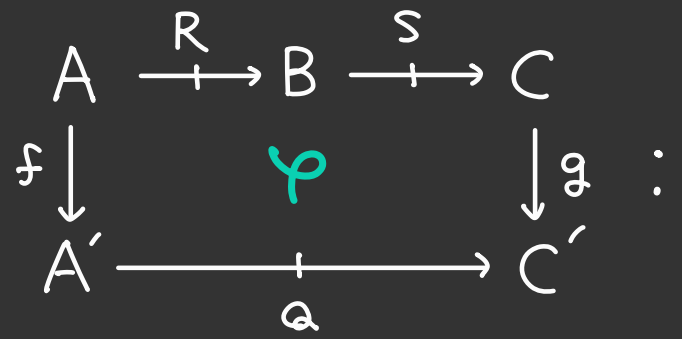
$$\begin{array}{c} \mathcal{E} \\ \mathbb{P}\downarrow \\ \mathcal{B} \end{array} : \text{cartesian fibration} \rightsquigarrow \mathbb{B}\mathrm{il}(\mathbb{P}) : \text{a VDC}$$

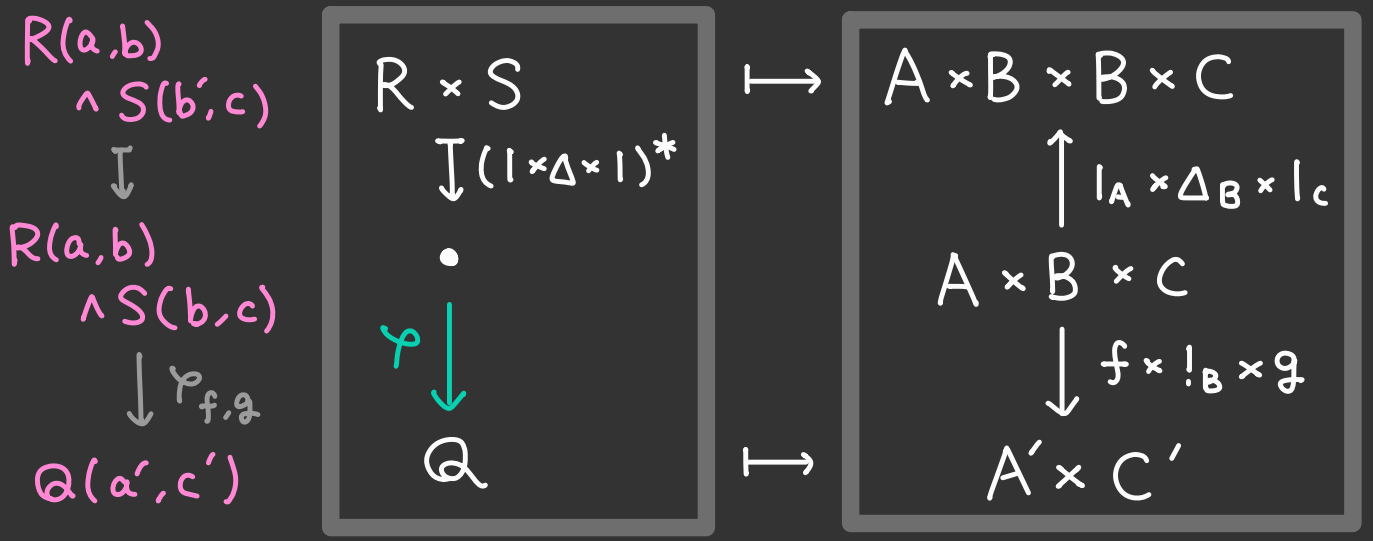
objects A : objects in $\mathcal{B}$

tight morphisms $A \xrightarrow{f} B$: morphisms in $\mathcal{B}$

loose morphisms $A \xrightarrow{R} B$: $R \mapsto_{\mathbb{P}} A \times B$

virtual cells





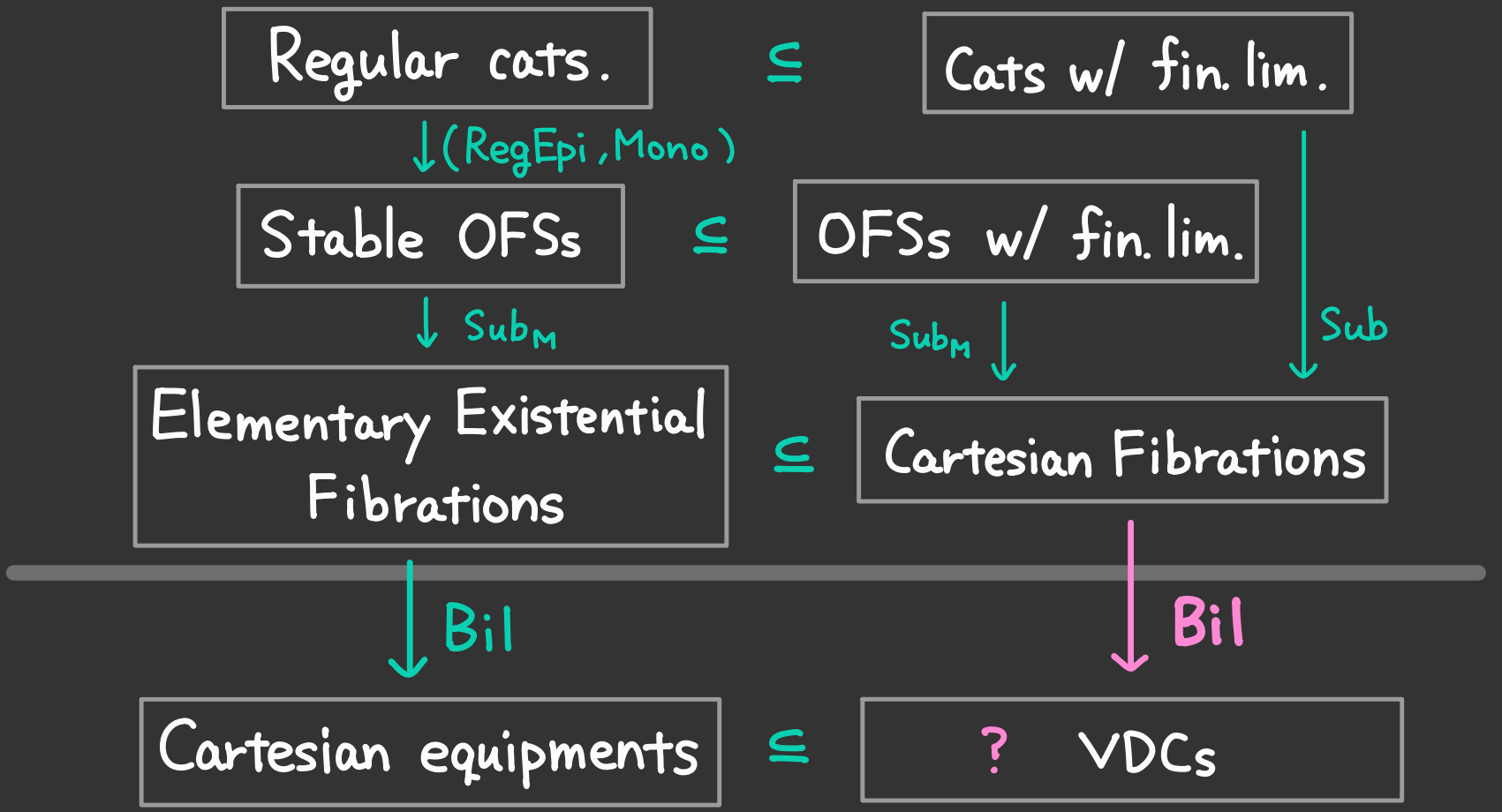
1. The construction $\mathbb{B}\mathrm{il}$

2. When is $\mathbb{B}\mathrm{il}(\mathbb{P})$ a cartesian equipment?

3. When does an equipment arise from $\mathbb{B}\mathrm{il}$?

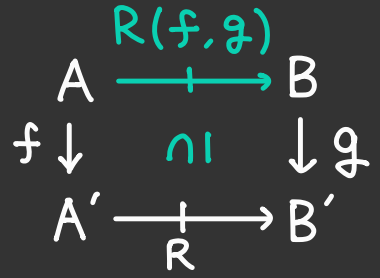
4. Double categorical logic

5. Ongoing work


Regular cats.
⊆
Cats w/ fin. lim.
↓(RegEpi, Mono)
Stable OFSs
⊆
OFSs w/ fin. lim.
↓ Sub_M
Sub_M ↓
Sub ↓
Elementary Existential Fibrations
⊆
Cartesian Fibrations
↓ Bil
Bil ↓
Cartesian equipments
⊆
? VDCs

$\mathbb{B}\mathrm{il}(\mathbb{P})$ has some nice structure.

- It has restrictions.



- It is cartesian.

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ \times & & \\ C & \xrightarrow[S]{} & D \end{array} = A \times C \xrightarrow{R \times S} B \times D$$

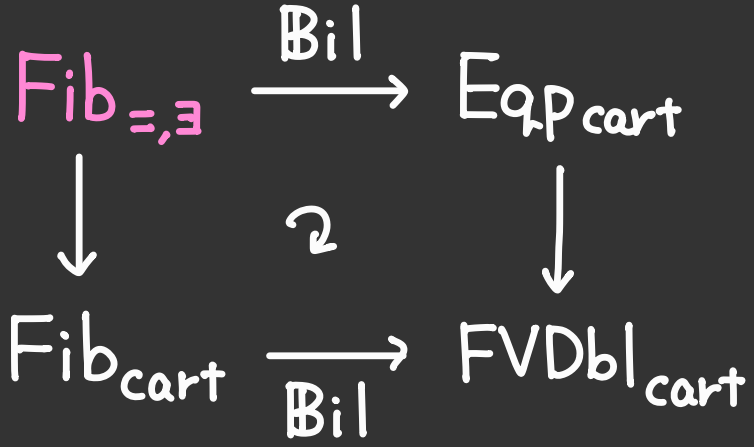
$$\mathbb{B}\mathrm{il}(\mathbb{P}) \underset{\top}{\overset{\Delta}{\rightleftarrows}} \mathbb{B}\mathrm{il}(\mathbb{P}) \times \mathbb{B}\mathrm{il}(\mathbb{P})$$

$$\mathbb{B}\mathrm{il}(\mathbb{P}) \underset{\top}{\overset{!}{\rightleftarrows}} 1$$

We call such a VDC a cartesian fibrational VDC.

Examples

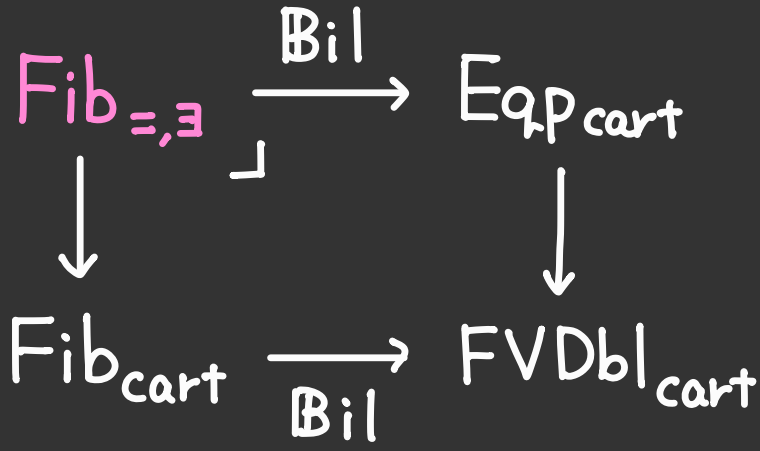
- $\mathcal{V}$: symmetric monoidal $\rightsquigarrow$ $\mathcal{V}$-$\mathbb{C}$AT is a CFVDC.
- A cartesian equipment underlies a CFVDC.



Theorem [N.] $\mathbb{P}$: cart. fibration

$\mathbb{P}$ is elementary existential $\iff$ $\mathbb{B}\mathrm{il}(\mathbb{P})$ is a cart. eqp.

In fact, we have a bipullback :



Examples

- $\mathcal{C}$: a cat. w/ fin. lim.

  $\mathbb{R}\mathrm{el}(\mathcal{C})$: cart. eqp. $\iff$ $\mathrm{Sub}(\mathcal{C})$ $\downarrow$ cod $\mathcal{C}$: elem. exst.

  [Jacobs '99] $\iff$ $\mathcal{C}$: regular

- $\mathcal{Q}$: a cat. w/ fin. prod.

  $\mathcal{Q}$-$\mathbb{R}\mathrm{el}$: cart. eqp. $\iff$ $\mathcal{Q}$-Fam $\downarrow$ Set : elem. exst.

  [N.] $\iff$ $\mathcal{Q}$ has distributive coproducts.

Technical remark

A cartesian fibration $p : \mathcal{E} \to \mathcal{B}$ is elementary existential if

(i) $\mathcal{E}_{A\times B} \underset{\pi^*}{\overset{\exists_\pi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}_A$, $\mathcal{E}_{A\times B} \underset{(1\times\Delta)^*}{\overset{\exists_{1\times\Delta}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}_{A\times B\times B}$.

(ii) the Beck-Chevalley condition for $\exists_\pi$, $\exists_{1\times\Delta}$.

compatibility of $\exists$ and $(-)^*$

$$\begin{array}{ccc} \mathcal{E}_{A\times B} & \xrightarrow{\exists_\pi} & \mathcal{E}_A \\ {\scriptstyle (f\times 1)^*}\downarrow & \Rightarrow & \downarrow{\scriptstyle f^*} \\ \mathcal{E}_{A'\times B} & \xrightarrow[\exists_\pi]{} & \mathcal{E}_{A'} \end{array} \quad , \quad \begin{array}{ccc} \mathcal{E}_{A\times B} & \xrightarrow{\exists_{1\times\Delta}} & \mathcal{E}_{A\times B\times B} \\ {\scriptstyle (f\times 1)^*}\downarrow & \Rightarrow & \downarrow{\scriptstyle (f\times 1\times 1)^*} \\ \mathcal{E}_{A'\times B} & \xrightarrow[\exists_{1\times\Delta}]{} & \mathcal{E}_{A'\times B\times B} \end{array}$$

(iii) Frobenius reciprocity for $\exists_\pi$, $\exists_{1\times\Delta}$.

compatibility of $\exists$ and $\wedge$

$$\begin{array}{ccc} \mathcal{E}_{A\times B}\times\mathcal{E}_A & \xrightarrow{\exists_\pi\times 1} & \mathcal{E}_A\times\mathcal{E}_A \\ {\scriptstyle \wedge\circ(1\times f^*)}\downarrow & \Rightarrow & \downarrow{\scriptstyle \wedge} \\ \mathcal{E}_{A\times B} & \xrightarrow[\exists_\pi]{} & \mathcal{E}_A \end{array} \quad , \quad \begin{array}{ccc} \mathcal{E}_{AB}\times\mathcal{E}_{ABB} & \xrightarrow{\exists_{1\times\Delta}\times 1} & \mathcal{E}_{ABB} \\ {\scriptstyle \wedge\circ(1\times f^*)}\downarrow & \Rightarrow & \downarrow{\scriptstyle \wedge} \\ \mathcal{E}_{AB} & \xrightarrow[\exists_{1\times\Delta}]{} & \mathcal{E}_{ABB} \end{array}$$

The last two conditions are encoded in the def. of cartesian equipments, mostly in the pseudo-ness of $\times$.

1. The construction $\mathbb{B}\mathrm{il}$

2. When is $\mathbb{B}\mathrm{il}(\mathbb{P})$ a cartesian equipment?

3. When does an equipment arise from $\mathbb{B}\mathrm{il}$?

4. Double categorical logic

5. Ongoing work

Characterizing double categories of relations

Regular categories

Stable factorization systems

Elementary existential fibrations

Bil ⟶

DCR [Lambert '22]

[Hoshino, N. '25]

Cartesian equipments

10.2. Theorem. *If $\mathbb{D}$ is a 'double category of relations' with a subobject comprehension scheme then the identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an adjoint equivalence*

$$\mathbb{R}\mathbf{el}(\mathbb{D}_0) \simeq \mathbb{D}.$$

In short, any 'double category of relations' with a subobject comprehension scheme is equivalent to a double category $\mathbb{R}\mathbf{el}(\mathscr{E})$ for some regular category $\mathscr{E}$.

Characterizing double categories of relations

Regular categories

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Bil ⟶

DCR [Lambert '22]

[Hoshino, N. '25]

Cartesian equipments

Theorem 3.3.20 *The following are equivalent for a double category* $\mathbb{D}$.

(i) $\mathbb{D}$ *is equivalent to* $\mathbb{R}\mathrm{el}_{E,M}(\boldsymbol{C})$ *for some category* $\boldsymbol{C}$ *with finite limits and a stable factorisation system* (E, M) *on* $\boldsymbol{C}$.

(ii) $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and an* M*-comprehension scheme for some stable system* M *on* $\boldsymbol{V}(\mathbb{D})$.

(iii) $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and a left-sided* M*-comprehension scheme for some stable system* M *on* $\boldsymbol{V}(\mathbb{D})$.

(iv) $\mathit{Fib}(\mathbb{D})$ *is closed under composition, and* $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and strong tabulators.*

(v) $\mathit{Fib}(\mathbb{D})$ *is closed under composition, and* $\mathbb{D}$ *is a cartesian equipment with Beck-Chevalley pullbacks and left-sided strong tabulators.*

Characterizing double categories of relations

Regular categories

Stable factorization systems

Elementary existential fibrations

$\mathbb{B}\mathrm{il}$ ⟶

DCR [Lambert '22]

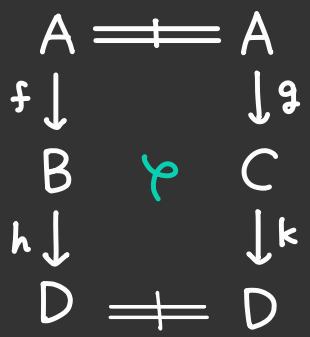
[Hoshino, N. '25]

Cartesian equipments

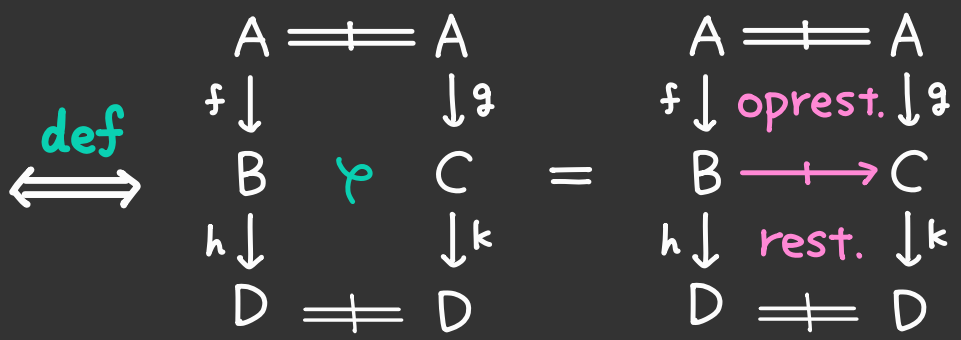
Q. $\mathbb{D} \simeq \mathbb{B}\mathrm{il}(\mathbb{P})$ $(\exists\, \mathbb{P} \in \mathrm{Fib}_{=,\exists})$

$\Longleftrightarrow$ $\mathbb{D}$ satisfies what ?

Beck-Chevalley in equipments



is Beck-Chevalley



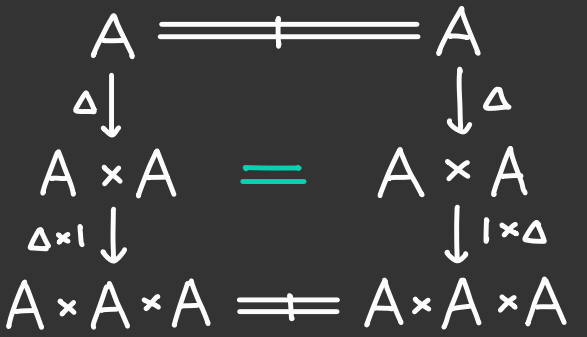
$\iff$ $f^* g_* \xrightarrow{\cong} h_* k^*$

[Walters, Wood '08]

The Frobenius axiom

$\mathbb{D}$: cart. eqp.

$A \in \mathbb{D}$ is Frobenius $\overset{\text{def}}{\Leftrightarrow}$



is B.C.

$\Leftrightarrow \quad \Delta^* \Delta_* \overset{\cong}{\Longrightarrow} (\Delta \times 1)_* (1 \times \Delta)^*$ [Walters, Wood '08]

Example $A \in \mathbb{CAT}$ is Frobenius $\Leftrightarrow A$: groupoid

A cart. eqp. is Frobenius if all objects are Frobenius.

Theorem [N.]

$\exists\, \mathbb{p} \in \mathrm{Fib}_{=,\exists}\quad \mathbb{D} \simeq \mathbb{B}\mathrm{il}(\mathbb{p}) \iff \mathbb{D}$ is Frobenius

In fact,

$$\mathrm{Fib}_{=,\exists} \xrightarrow[\simeq_{bi}]{\mathbb{B}\mathrm{il}} \mathrm{Eqp}_{\mathrm{Frob}} \subseteq \mathrm{Eqp}_{\mathrm{cart}}$$

Proof idea



induce self-duality on A [Carboni, Walters '87]

1. The construction $\mathbb{B}\mathrm{il}$

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4. Double categorical logic

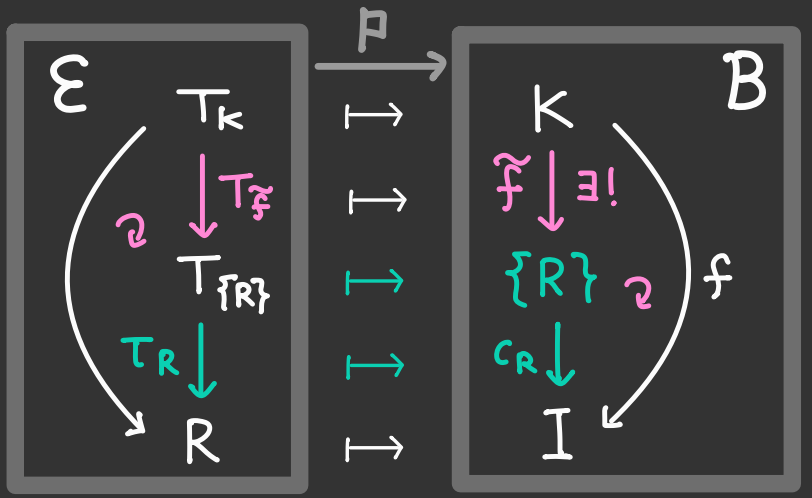
5. Ongoing work

Logic	Fibrations	VDCs
Regular logic	Elementary existential fibrations	Frobenius cartesian equipments
$\wedge, \top, =, \exists$		
Axioms Principles	Structures on fibrations	Structures on VDCs
$\dfrac{x:I \vdash \varphi(x)\ \mathrm{Prop}}{\{x:I \mid \varphi(x)\}}$	Comprehension structure	?

[Lawvere '69, Ehrhard '88]

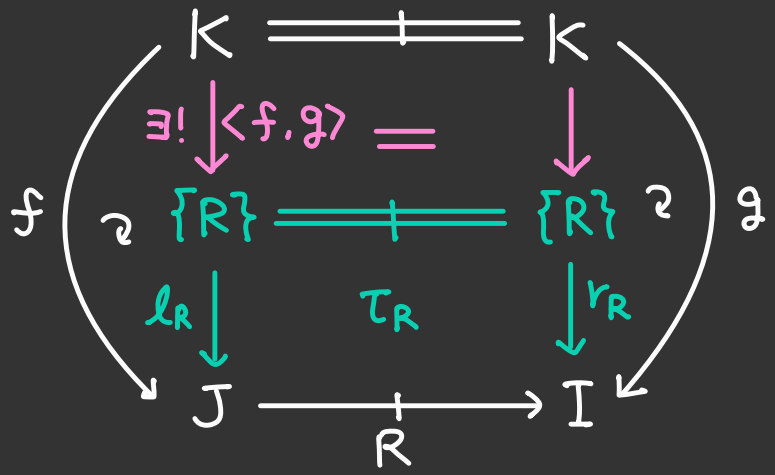
Fibrations

A comprehension $\{R\}$ of $R \in \mathcal{E}_I$ is defined by :



Double categories

A tabulator $[R]$ of $J \xrightarrow{R} I$ is defined by :



Proposition [N.] $p : \mathcal{E} \to \mathcal{B} \in \mathrm{Fib}_{=,\exists}$

Every $R \in \mathcal{E}$ has a comprehension.

$\Updownarrow$

Every $J \overset{R}{\nrightarrow} I$ in $\mathbb{B}\mathrm{il}(p)$ has a tabulator.

(Fullness of a comprehension corresponds to effectiveness of a tabulator.)

Logic

Function extensionality

$$\forall x \quad f(x) = g(x) \implies f = g$$

Uniqueness of identity proofs

$$\varphi, \gamma : s = t \implies \varphi = \gamma$$

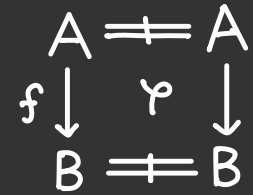
Unique choice principle

$R(x, y)$: total, single-valued

$$\implies \exists f \quad R(x, y) \equiv y = f(x)$$

Double categories

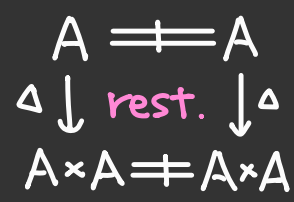
Unit-purity [Aleiferi '18]



$$\implies \quad f = g, \quad \varphi = \mathrm{id}_f$$

Separability

[Lack, Walters, Wood '10]

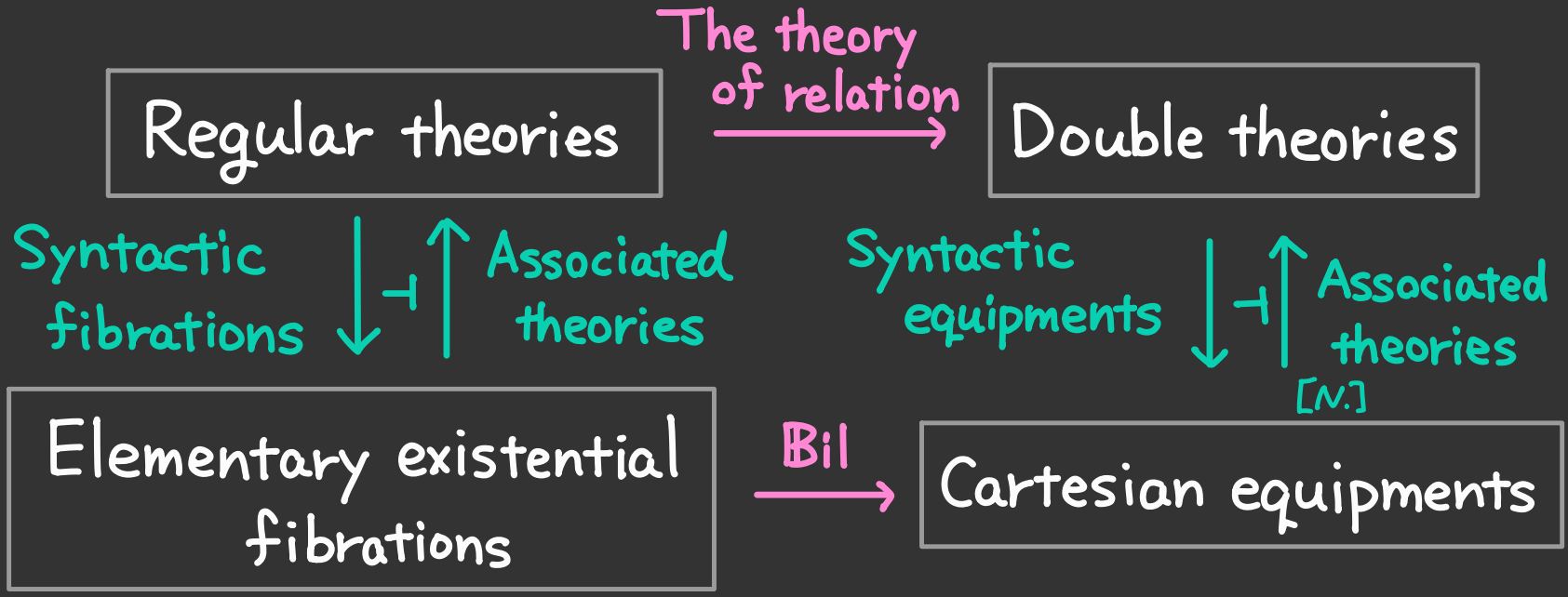


Map-completeness

(Cauchy [Paré '21])



Double functorial semantics



COHERENT THEORIES AS DOUBLE LAWVERE THEORIES

[Paré '09]

1. The construction $\mathbb{B}\mathrm{il}$

2. When is $\mathbb{B}\mathrm{il}(\mathbb{P})$ a cartesian equipment?

3. When does an equipment arise from $\mathbb{B}\mathrm{il}$?

4. Double categorical logic

5. Ongoing work

joint with Aaron Fairbanks

Further questions

- How about monoidal fibrations ? (cf. [Shulman '12])
- Elementary / existential fibrations, separately?
- Logical completions in terms of double categories ?

⋮

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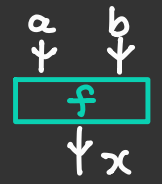
⋮

Why only binary relations ?

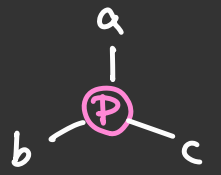
Do virtual double categories capture what we want ?

Def A virtual cartesian double category consists of:

(1) A (fat/unbiased) cartesian multicategory of objects and tight morphisms.

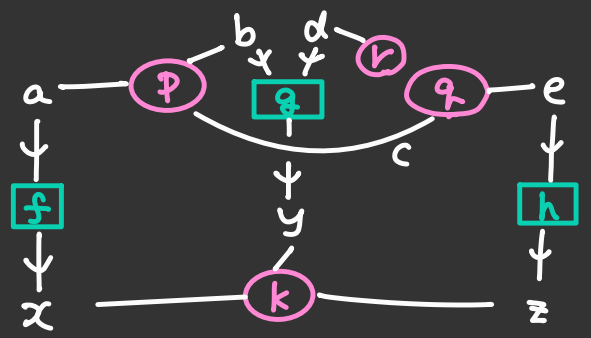


(2) Loose cells depending on finite families of objects.



(3) A family of sets of 2-cells depending on frames.

(4) Composition of 2-cells



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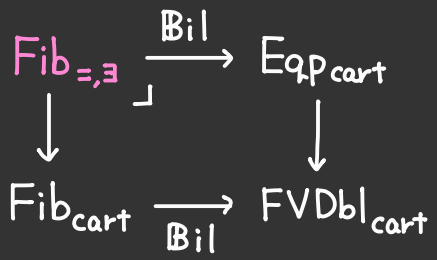
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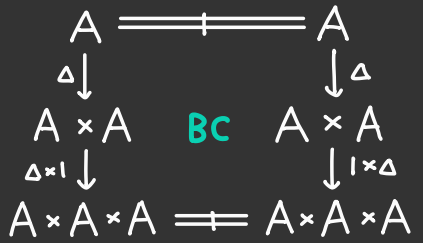
Thank you!

I updated the preprint as a standalone paper.



2501.17869





Things I omitted

- Regular fibrations $\simeq$ Cartesian equipments with BC pullbacks

  (Corollary 4.5)

- Recovering the characterization of $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ (§4.2)
- Map-completion & cartesian bicategories

$$\mathbf{Fib}_{\times\wedge=\exists,\mathbf{m\text{-}disc}} \underset{\mathbf{uni}}{\overset{\mathbb{B}\mathrm{il}}{\underset{\leftarrow}{\overset{\rightarrow}{\simeq}}}} \mathbf{Eqp}_{\mathbf{Frob},\mathbf{m\text{-}disc}} \underset{\mathbb{M}\mathrm{ap}(-)}{\overset{\mathbf{L}(-)}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathbf{CartBi}_{\mathbf{Frob},\mathbf{m\text{-}disc}}$$

(§4.6)